Theory of the Lattice Boltzmann Method

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B. D. and A. J. C. Ladd, arXiv:0803.2826v2,
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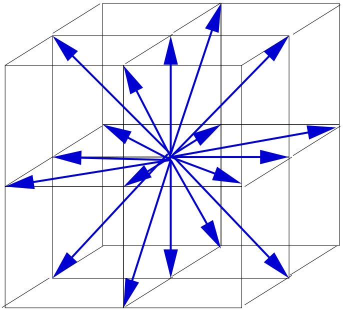
Lattice Boltzmann



- linearized Boltzmann equation (kinetic theory of gases)
- fully discretized
- sites $\vec{r}$, lattice spacing a
- time t, time step h

- $\vec{c}_i$ small set of velocities
- $\vec{c}_i h$ connects two sites
- $n_i(\vec{r}, t)$: real number, mass density on site $\vec{r}$ corresponding to velocity $\vec{c}_i$

$$n_i(\vec{r} + \vec{c}_i h, t + h) = n_i^\star(\vec{r}, t) = n_i(\vec{r}, t) + \Delta_i(\vec{r}, t)$$

Conservation laws, symmetries

$$n_i(\vec{r} + \vec{c}_i h, t + h) = n_i^\star(\vec{r}, t) = n_i(\vec{r}, t) + \Delta_i \left\{ n_i(\vec{r}, t) \right\}$$

$$\rho = \sum_i n_i$$

$$\vec{j} = \rho \vec{u} = \sum_i n_i \vec{c}_i$$

$$\sum_i \Delta_i = \sum_i \Delta_i \vec{c}_i = 0$$

- 👍 mass conservation
- 👍 momentum conservation
- 👍 locality
- 👎 rotational symmetry (lattice!)
- 👎 Galilei invariance (finite number of velocities)

Low Mach number physics

- only $u \ll c_i$
- only $u \ll c_s$
- $Ma = u/c_s \ll 1$
- low Mach number $\Rightarrow$ compressibility does not matter $\Rightarrow$ equation of state does not matter $\Rightarrow$ choose ideal gas!

m_p particle mass:

$$p = \frac{\rho}{m_p} k_B T$$

$$c_s^2 = \frac{\partial p}{\partial \rho} = \frac{1}{m_p} k_B T$$

$$p = \rho c_s^2$$

$$k_B T = m_p c_s^2$$

Where we want to get

in the continuum limit $a \to 0$, $h \to 0$:

$$\partial_t \rho + \partial_\alpha j_\alpha = 0$$

$$\partial_t j_\alpha + \partial_\beta \left(\rho c_s^2 \delta_{\alpha\beta} + \rho u_\alpha u_\beta \right) = \partial_\beta \sigma_{\alpha\beta}$$

$$\sigma_{\alpha\beta} = \eta_{\alpha\beta\gamma\delta} \partial_\gamma u_\delta$$

$$\eta_{\alpha\beta\gamma\delta} = \left(\zeta - \frac{2}{3}\eta \right) \delta_{\alpha\beta}\delta_{\gamma\delta} + \eta \left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} \right)$$

- η shear viscosity
- ζ bulk viscosity

Difference equation $\longrightarrow$ differential equation

Example: consider

$$f(x+a) - f(x) = g(x)$$

small a:

$$a \frac{d}{dx} f(x) \approx g(x)$$

continuum limit $a \to 0$:

$$\frac{d}{dx} f(x) = \lim_{a \to 0} \frac{g(x)}{a}$$

watch out: if the rhs does not exist, then the continuum limit does not exist!

corrections to the leading behavior?

$$f(x+a) - f(x) = g(x)$$

set

$$\begin{aligned} D &= a\frac{d}{dx} \\ g(x) &= a\left(g_0(x) + a g_1(x) + a^2 g_2(x) + \ldots\right) \end{aligned}$$

g_i independent of a

$$\begin{aligned} f(x_0+a) &= f(x_0) + a \left.\frac{df}{dx}\right|_{x=x_0} + \frac{a^2}{2!} \left.\frac{d^2 f}{dx^2}\right|_{x=x_0} + \ldots \\ &= \exp\left(a\frac{d}{dx}\right) f(x)\Big|_{x=x_0} = \exp(D)\, f(x)|_{x=x_0} \end{aligned}$$

$$\begin{aligned} \Rightarrow [\exp(D) - 1]\, f(x) &= g(x) \\ f(x) &= [\exp(D) - 1]^{-1}\, g(x) \\ Df(x) &= D\,[\exp(D) - 1]^{-1}\, g(x) \end{aligned}$$

now,

$$\frac{x}{\exp(x)-1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - + \ldots = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

Bernoulli numbers B_k

$$\begin{aligned}
f(x+a) - f(x) &= g(x) \Leftrightarrow \\
Df(x) &= \left[1 - \frac{D}{2} + \frac{D^2}{12} + \ldots\right] g(x) \\
\frac{d}{dx} f(x) &= \left[1 - \frac{D}{2} + \frac{D^2}{12} + \ldots\right] \\
&\quad \left[g_0(x) + a g_1(x) + a^2 g_2(x) + \ldots\right]
\end{aligned}$$

systematic expansion in powers of a!

LB continuum limit: how??

$a \to 0$, $h \to 0$ to be replaced by $\varepsilon \to 0$:

- wave–like scaling: $a/h = const.$

$$
\begin{aligned}
a &= \varepsilon a_0 \\
h &= \varepsilon h_0 \\
\vec{c}_i &= \vec{c}_{i0} \\
n_i(\vec{r} + \varepsilon \vec{c}_{i0} h_0, t + \varepsilon h_0) - n_i(\vec{r}, t) &= \Delta_i
\end{aligned}
$$

- diffusive scaling: $a^2/h = const.$

$$
\begin{aligned}
a &= \varepsilon a_0 \\
h &= \varepsilon^2 h_0 \\
\vec{c}_i &= \varepsilon^{-1} \vec{c}_{i0} \\
n_i(\vec{r} + \varepsilon \vec{c}_{i0} h_0, t + \varepsilon^2 h_0) - n_i(\vec{r}, t) &= \Delta_i
\end{aligned}
$$

Expanding the solution

lhs is $O(\varepsilon) \Rightarrow$ rhs must be $O(\varepsilon)$!
ansatz:

$$
\begin{aligned}
n_i &= n_i^{(0)} + \varepsilon n_i^{(1)} + O(\varepsilon^2) \\
\Delta_i \{n_i\} &= \Delta_i^{(0)} + \varepsilon \Delta_i^{(1)} + O(\varepsilon^2) \\
&= \Delta_i \left\{ n_i^{(0)} \right\} + \varepsilon \sum_j \left. \frac{\partial \Delta_i}{\partial n_j} \right|_{n_i = n_i^{(0)}} n_j^{(1)} + O(\varepsilon^2) \\
&=: \varepsilon \sum_j L_{ij} n_j^{(1)} + O(\varepsilon^2)
\end{aligned}
$$

and

$$
\Delta_i \left\{ n_i^{(0)} \right\} = 0
$$

conservation laws:

$$
\sum_i \Delta_i^{(k)} = \sum_i \Delta_i^{(k)} \vec{c}_i = 0
$$

Zeroth order

$$0 = \Delta_i^{(0)} = \Delta_i \{ n_i^{(0)} \}$$

- $\{ n_i^{(0)} \}$ collisional invariant, $\{ n_i^{(0)} \} = n_i^{eq}$
- no spurious conservation laws $\Rightarrow$
- $n_i^{(0)} = n_i^{(0)}(\rho, \vec{j})$

No expansion for conserved quantities!

$$
\begin{aligned}
n_i &= n_i^{(0)} + \varepsilon n_i^{(1)} + O(\varepsilon^2) \\
\rho &= \rho^{(0)} + \varepsilon \rho^{(1)} + O(\varepsilon^2) \\
\vec{j} &= \vec{j}^{(0)} + \varepsilon \vec{j}^{(1)} + O(\varepsilon^2) \\
n_i^{(0)} &= n_i^{(0)} \left(\rho, \vec{j} \right) \\
&= n_i^{(0)} \left(\rho^{(0)} + \varepsilon \rho^{(1)} + O(\varepsilon^2), \vec{j}^{(0)} + \varepsilon \vec{j}^{(1)} + O(\varepsilon^2) \right)
\end{aligned}
$$

no expansion for $n_i^{(0)} \Rightarrow$

$$
\begin{aligned}
\rho^{(1)} = \rho^{(2)} = \ldots &= 0 \\
\vec{j}^{(1)} = \vec{j}^{(2)} = \ldots &= 0
\end{aligned}
$$

i. e.

- $$\rho^{(0)} = \rho$$
- $$\vec{j}^{(0)} = \vec{j}$$
- $$\sum_i n_i^{eq} = \rho$$
- $$\sum_i n_i^{eq} \vec{c}_i = \vec{j}$$

Velocity moments

- mass density

$$\rho = \sum_i n_i$$

- momentum density

$$j_\alpha = \sum_i n_i c_{i\alpha}$$

- stress

$$\Pi_{\alpha\beta} = \sum_i n_i c_{i\alpha} c_{i\beta}$$

- 3rd moment

$$\Phi_{\alpha\beta\gamma} = \sum_i n_i c_{i\alpha} c_{i\beta} c_{i\gamma}$$

LB continuum limit: wave–like scaling

$$n_i(\vec{r} + \varepsilon \vec{c}_i h_0, t + \varepsilon h_0) - n_i(\vec{r}, t) = \Delta_i$$

set

$$D_i = \varepsilon h_0 \partial_t + \varepsilon h_0 c_{i\alpha} \partial_\alpha$$

$$D_i n_i = \left[1 - \frac{D_i}{2} + \frac{D_i^2}{12} + \dots \right] \Delta_i$$

- ε^1:

$$\left(h_0 \partial_t + h_0 c_{i\alpha} \partial_\alpha \right) n_i^{(0)} = \Delta_i^{(1)}$$

- ε^2:

$$\left(h_0 \partial_t + h_0 c_{i\alpha} \partial_\alpha \right) n_i^{(1)} = \Delta_i^{(2)} - \frac{1}{2} \left(h_0 \partial_t + h_0 c_{i\alpha} \partial_\alpha \right) \Delta_i^{(1)}$$

take zeroth and first velocity moment:

- ε^1:

$$\begin{aligned} h_0 \partial_t \rho + h_0 \partial_\alpha j_\alpha &= 0 \\ h_0 \partial_t j_\beta + h_0 \partial_\alpha \Pi^{(0)}_{\alpha\beta} &= 0 \end{aligned}$$

or

$$\begin{aligned} \partial_t \rho + \partial_\alpha j_\alpha &= 0 \\ \partial_t j_\beta + \partial_\alpha \Pi^{(0)}_{\alpha\beta} &= 0 \end{aligned}$$

- ε^2:

$$\begin{aligned} 0 &= 0 \\ h_0 \partial_\alpha \Pi^{(1)}_{\alpha\beta} &= -\frac{1}{2} h_0 \partial_\alpha \left(\Pi^{(1)\star}_{\alpha\beta} - \Pi^{(1)}_{\alpha\beta} \right) \end{aligned}$$

or

$$\frac{1}{2} \partial_\alpha \left(\Pi^{(1)\star}_{\alpha\beta} + \Pi^{(1)}_{\alpha\beta} \right) = 0$$

consequences:

- $\Pi^{(0)}$ depends only on ρ, $\vec{j}$ (locally!!)
- $\Rightarrow$ $\Pi^{(0)}$ must be the Euler stress!
- i. e.

$$\sum_i n_i^{eq} c_{i\alpha} c_{i\beta} = \rho c_s^2 \delta_{\alpha\beta} + \rho u_\alpha u_\beta$$

- stress relaxation at order ε^2 gives rise to “some sort of” dissipation, **but no relation** to the previous order
- i. e. relation to velocity gradients (viscous shear stresses) can not be established!
- we find: Euler equations at order ε^1, but no useful results beyond!

can we at least adjust n_i^{eq} such that we get Euler in the leading order? **YES!**

The equilibrium populations

ansatz (Euler stress is a 2nd order polynomial in $\vec{u}$!):

$$n_i^{eq}(\rho, \vec{u}) = w_i \rho \left(1 + A\vec{u} \cdot \vec{c}_i + B(\vec{u} \cdot \vec{c}_i)^2 + Cu^2\right)$$

w_i positive weights, identical within a shell. cubic symmetry:

$$\sum_i w_i = 1$$

$$\sum_i w_i c_{i\alpha} = 0$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} = \sigma_2 \, \delta_{\alpha\beta}$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} = 0$$

$$\sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta} = \kappa_4 \, \delta_{\alpha\beta\gamma\delta}$$

$$+\sigma_4 \left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)$$

mass:

$$
\begin{aligned}
\rho &= \sum_i n_i^{eq} \\
&= \rho \sum_i w_i \left(1 + A\vec{u} \cdot \vec{c}_i + B(\vec{u} \cdot \vec{c}_i)^2 + Cu^2\right) \\
0 &= B u_\alpha u_\beta \sum_i w_i c_{i\alpha} c_{i\beta} + Cu^2 \\
&= B u_\alpha u_\beta \sigma_2 \delta_{\alpha\beta} + Cu^2 \\
&= \left(B\sigma_2 + C\right) u^2 \\
C + B\sigma_2 &= 0
\end{aligned}
$$

momentum:

$$
\begin{aligned}
\rho u_\alpha &= \sum_i n_i^{eq} c_{i\alpha} \\
&= \rho \sum_i w_i c_{i\alpha} \left(1 + A\vec{u}\cdot\vec{c}_i + B(\vec{u}\cdot\vec{c}_i)^2 + Cu^2\right) \\
&= \rho A u_\beta \sum_i w_i c_{i\alpha} c_{i\beta} \\
&= \rho A u_\beta \sigma_2 \delta_{\alpha\beta} \\
&= \rho A \sigma_2 u_\alpha \\
A\sigma_2 &= 1
\end{aligned}
$$

stress:

$$
\begin{aligned}
c_s^2 \delta_{\alpha\beta} + u_\alpha u_\beta &= \frac{1}{\rho} \sum_i n_i^{eq} c_{i\alpha} c_{i\beta} \\
&= \sum_i w_i c_{i\alpha} c_{i\beta} \left(1 + A \vec{u} \cdot \vec{c}_i + B (\vec{u} \cdot \vec{c}_i)^2 + C u^2 \right) \\
&= \left(1 + C u^2 \right) \sigma_2 \delta_{\alpha\beta} + B u_\gamma u_\delta \kappa_4 \delta_{\alpha\beta\gamma\delta} \\
&+ B u_\gamma u_\delta \sigma_4 \left(\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma} \right)
\end{aligned}
$$

hence $\kappa_4 = 0$ and

$$
c_s^2 \delta_{\alpha\beta} + u_\alpha u_\beta = \left(\sigma_2 + C \sigma_2 u^2 + B \sigma_4 u^2 \right) \delta_{\alpha\beta} + 2 B \sigma_4 u_\alpha u_\beta
$$

i. e. $\sigma_2 = c_s^2$, $C \sigma_2 + B \sigma_4 = 0$, $2 B \sigma_4 = 1$

taken all together:

$$
\begin{aligned}
\kappa_4 &= 0 \\
\sigma_2 &= c_s^2 \\
2B\sigma_4 &= 1 \\
C\sigma_2 + B\sigma_4 &= 0 \\
C + B\sigma_2 &= 0 \\
A\sigma_2 &= 1
\end{aligned}
$$

six equations, six unknowns. multiply Eq. 5 with σ_2 and compare with Eq. 4. hence the solution is:

$$
\begin{aligned}
\kappa_4 &= 0 \\
\sigma_2 &= c_s^2 \\
\sigma_4 &= c_s^4 \\
A &= 1/c_s^2 \\
B &= 1/(2c_s^4) \\
C &= -1/(2c_s^2)
\end{aligned}
$$

form of the equilibrium populations is

$$n_i^{eq}(\rho, \vec{u}) = w_i \rho \left(1 + \frac{\vec{u} \cdot \vec{c}_i}{c_s^2} + \frac{(\vec{u} \cdot \vec{c}_i)^2}{2 c_s^4} - \frac{u^2}{2 c_s^2} \right)$$

what are the weights? we need to satisfy the three conditions:

$$\begin{aligned} \sum_i w_i &= 1 \\ \kappa_4 &= 0 \\ \sigma_4 &= \sigma_2^2 \end{aligned}$$

therefore, at least three shells are needed! each shell is assigned its own σ_2, σ_4, κ_4 (assuming weight one).

D3Q19

- one zero velocity: $\vec{c}_i = 0$, weight w_0
- six nearest neighbors: $\vec{c}_i = (a/h)(\pm 1, 0, 0)$, $(a/h)(0, \pm 1, 0)$, $(a/h)(0, 0, \pm 1)$, weight w_I
- twelve next–nearest neighbors: $\vec{c}_i = (a/h)(\pm 1, \pm 1, 0)$, $(a/h)(\pm 1, 0, \pm 1)$, $(a/h)(0, \pm 1, \pm 1)$, weight w_{II}
- zeroth shell: velocity moments trivial

- first shell:

$$
\begin{aligned}
\sum_i c_{i1}^2 &= 2(a/h)^2 = \sigma_2(l) \\
\sum_i c_{i1}^4 &= 2(a/h)^4 = \kappa_4(l) + 3\sigma_4(l) \\
\sum_i c_{i1}^2 c_{i2}^2 &= 0 = \sigma_4(l) \\
\sigma_2(l) &= 2(a/h)^2 \\
\sigma_4(l) &= 0 \\
\kappa_4(l) &= 2(a/h)^4
\end{aligned}
$$

- second shell:

$$
\begin{aligned}
\sum_i c_{i1}^2 &= 8(a/h)^2 = \sigma_2(II) \\
\sum_i c_{i1}^4 &= 8(a/h)^4 = \kappa_4(II) + 3\sigma_4(II) \\
\sum_i c_{i1}^2 c_{i2}^2 &= 4(a/h)^4 = \sigma_4(II) \\
\sigma_2(II) &= 8(a/h)^2 \\
\sigma_4(II) &= 4(a/h)^4 \\
\kappa_4(II) &= -4(a/h)^4
\end{aligned}
$$

$$
\begin{aligned}
0 &= \kappa_4 \\
&= w_I \kappa_4(I) + w_{II} \kappa_4(II) \\
&= 2 w_I - 4 w_{II} \\
w_I &= 2 w_{II} \\
\sigma_2 &= w_I \sigma_2(I) + w_{II} \sigma_2(II) \\
&= w_{II} \left(2 \sigma_2(I) + \sigma_2(II) \right) \\
&= w_{II} (a/h)^2 \left(2 \cdot 2 + 8 \right) \\
&= 12 w_{II} (a/h)^2 \\
\sigma_4 &= w_I \sigma_4(I) + w_{II} \sigma_4(II) \\
&= w_{II} \left(2 \sigma_4(I) + \sigma_4(II) \right) \\
&= 4 w_{II} (a/h)^4 \\
&= \sigma_2^2 \\
&= 144 w_{II}^2 (a/h)^4
\end{aligned}
$$

$$
\begin{aligned}
1 &= 36 w_{II} \\
w_{II} &= \frac{1}{36} \\
w_I &= 2 w_{II} = \frac{1}{18} \\
1 &= w_0 + 6 w_I + 12 w_{II} \\
&= w_0 + \frac{1}{3} + \frac{1}{3} \\
w_0 &= \frac{1}{3} \\
c_s^2 &= \sigma_2 \\
&= 12 w_{II} (a/h)^2 \\
&= \frac{1}{3} (a/h)^2
\end{aligned}
$$

all coefficients of n_i^{eq} known!

LB continuum limit: diffusive scaling

$$n_i(\vec{r} + \varepsilon \vec{c}_{i0} h_0, t + \varepsilon^2 h_0) - n_i(\vec{r}, t) = \Delta_i$$

watch out: $\vec{c}_i = \varepsilon^{-1} \vec{c}_{i0}$! define moments wrt $\vec{c}_{i0}$, not $\vec{c}_i$! e. g. $\vec{j} = \sum_i n_i \vec{c}_{i0}$ etc.!

set:

$$D_i = \varepsilon^2 h_0 \partial_t + \varepsilon h_0 c_{i0\alpha} \partial_\alpha$$

$$D_i n_i = \left[1 - \frac{D_i}{2} + \frac{D_i^2}{12} + \dots \right] \Delta_i$$

- ε^1:

$$h_0 c_{i0\alpha} \partial_\alpha n_i^{(0)} = \Delta_i^{(1)}$$

- ε^2:

$$h_0 \partial_t n_i^{(0)} + h_0 c_{i0\alpha} \partial_\alpha n_i^{(1)} = \Delta_i^{(2)} - \frac{1}{2} h_0 c_{i0\alpha} \partial_\alpha \Delta_i^{(1)}$$

1st order

$$h_0 c_{i0\alpha} \partial_\alpha n_i^{(0)} = \Delta_i^{(1)}$$

zeroth velocity moment:

$$\partial_\alpha j_\alpha = 0$$

first velocity moment:

$$\partial_\alpha \Pi_{\alpha\beta}^{(0)} = 0$$

2nd order

$$h_0 \partial_t n_i^{(0)} + h_0 c_{i0\alpha} \partial_\alpha n_i^{(1)} = \Delta_i^{(2)} - \frac{1}{2} h_0 c_{i0\alpha} \partial_\alpha \Delta_i^{(1)}$$

zeroth velocity moment:

$$\partial_t \rho = 0$$

incompressible fluid!

1st velocity moment:

$$\begin{aligned} \partial_t j_\beta + \partial_\alpha \Pi_{\alpha\beta}^{(1)} &= -\frac{1}{2} \partial_\alpha \left(\Pi_{\alpha\beta}^{(1)\star} - \Pi_{\alpha\beta}^{(1)} \right) \\ \partial_t j_\beta + \frac{1}{2} \partial_\alpha \left(\Pi_{\alpha\beta}^{(1)\star} + \Pi_{\alpha\beta}^{(1)} \right) &= 0 \end{aligned}$$

Adding the equations

$$\begin{aligned} \partial_t \rho + \partial_\alpha j_\alpha &= 0 \\ \partial_t j_\beta + \partial_\alpha \Pi^{(0)}_{\alpha\beta} + \frac{1}{2} \partial_\alpha \left(\Pi^{(1)\star}_{\alpha\beta} + \Pi^{(1)}_{\alpha\beta} \right) &= 0 \end{aligned}$$

looks like Navier–Stokes; $\Pi^{(0)}$ Euler stress, $(1/2)\left(\Pi^{(1)\star} + \Pi^{(1)}\right)$ viscous stress; **BUT**

- dynamics with *constraints*:

$$\begin{aligned} \partial_\alpha j_\alpha &= 0 \\ \partial_\alpha \Pi^{(0)}_{\alpha\beta} &= 0 \\ \partial_t \rho &= 0 \end{aligned}$$

- incompressible $\rightarrow$ pressure as a Lagrange multiplier
- difficult to analyze! (Junk / Luo / Klar)

All these difficulties go away when one combines wave–like and diffusive scaling in a multiple time scale analysis!

So, what is this?

The idea of multiple time scale expansion

example: damped oscillator

$$\frac{d^2}{dt^2}x + \frac{1}{\tau}\frac{d}{dt}x + \frac{1}{T^2}x = 0$$

- T oscillation period
- τ frictional relaxation time
- consider $T \ll \tau$ (weak damping)
- try to treat

  $$\varepsilon := \frac{T}{2\tau}$$

  as a small parameter for **perturbation theory**
- unit system: set $T = 1$

$$\frac{d^2}{dt^2}x + 2\varepsilon\frac{d}{dt}x + x = 0$$

$$\frac{d^2}{dt^2}x + 2\varepsilon \frac{d}{dt}x + x = 0$$

- $x(t = 0) = 1$, $\dot{x}(t = 0) = -\varepsilon$
- exactly solvable

$$x(t) = \exp\left(-\varepsilon t\right) \cos\left(\sqrt{1 - \varepsilon^2}\, t\right)$$

ε dependence looks harmless, **but** ...

Naive perturbation theory

$$\frac{d^2}{dt^2} x + 2\varepsilon \frac{d}{dt} x + x = 0$$

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$$

yields hierarchy:

$$\ddot{x}_k + x_k = -2\dot{x}_{k-1}$$

with (def.) $x_{-1} = 0$, plus corresponding hierarchy of initial conditions

- ε^0:

$$x_0 = \cos t$$

- ε^1:

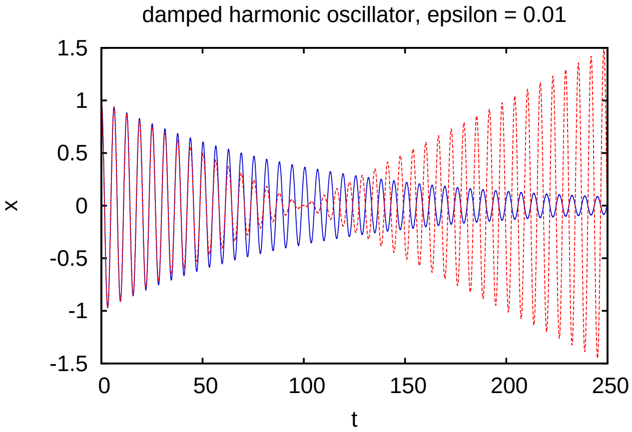
$$x_1 = -t \cos t$$

i. e., 1st order perturbation theory yields

$$x(t) = (1 - \varepsilon t) \cos t + O(\varepsilon^2)$$

identical with Taylor expansion of the exact solution!

For $t \gg 1/\varepsilon$ this becomes completely useless!!!



Deficiencies of naive perturbation theory:

- does not capture the presence of different time scales (here: fast oscillations vs. slow damping)
- typically, this occurs if one has *qualitatively different* behavior for $\varepsilon = 0$ and small $\varepsilon > 0$ (here: conservative vs. dissipative)
- "singular perturbation theory" needed

Multiple time scale analysis

Idea:

$$\begin{aligned} x(t) &= \exp(-\varepsilon t) \cos\left(\sqrt{1-\varepsilon^2} t\right) \\ &\approx \exp(-\varepsilon t) \cos t \\ &= \exp(-t_1) \cos t \\ &= x(t, t_1) \end{aligned}$$

with

$$t_1 = \varepsilon t$$

consider x as a function of *two independent* variables t, t_1
$\Rightarrow$ should be able to grasp the time scale separation!
hence, study expansion

$$x(t, t_1) = x_0(t, t_1) + \varepsilon x_1(t, t_1) + \varepsilon^2 x_2(t, t_1) + \ldots$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial t_1}{\partial t} \frac{\partial}{\partial t_1} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial t_1}$$

again the damped oscillator:

$$\begin{aligned} \frac{d}{dt} x &= p \\ \frac{d}{dt} p &= -2 \varepsilon p - x \end{aligned}$$

(expand both x and p)

ε^0:

$$\begin{aligned} \frac{\partial}{\partial t} x_0 &= p_0 \\ \frac{\partial}{\partial t} p_0 &= -x_0 \\ x_0 &= A(t_1) \cos t + B(t_1) \sin t \\ p_0 &= -A(t_1) \sin t + B(t_1) \cos t \end{aligned}$$

$A(t_1)$, $B(t_1)$ not yet known

ε^1:

$$
\begin{aligned}
\frac{\partial}{\partial t} x_1 &= p_1 - \frac{\partial}{\partial t_1} x_0 \\
\frac{\partial}{\partial t} p_1 &= -x_1 - \frac{\partial}{\partial t_1} p_0 - 2 p_0
\end{aligned}
$$

ansatz

$$
\begin{aligned}
x_1 &= C(t, t_1) \cos t + D(t, t_1) \sin t \\
p_1 &= -C(t, t_1) \sin t + D(t, t_1) \cos t
\end{aligned}
$$

yields

$$
\begin{aligned}
\frac{\partial C}{\partial t} &= -\frac{\partial A}{\partial t_1} - 2A \sin^2 t + 2B \sin t \cos t \\
\frac{\partial D}{\partial t} &= -\frac{\partial B}{\partial t_1} - 2B \cos^2 t + 2A \sin t \cos t
\end{aligned}
$$

integrate wrt t, but the solution **should not explode!!!**

now,

$$\left\langle \sin^2 t \right\rangle = \left\langle \cos^2 t \right\rangle = \frac{1}{2}$$

hence

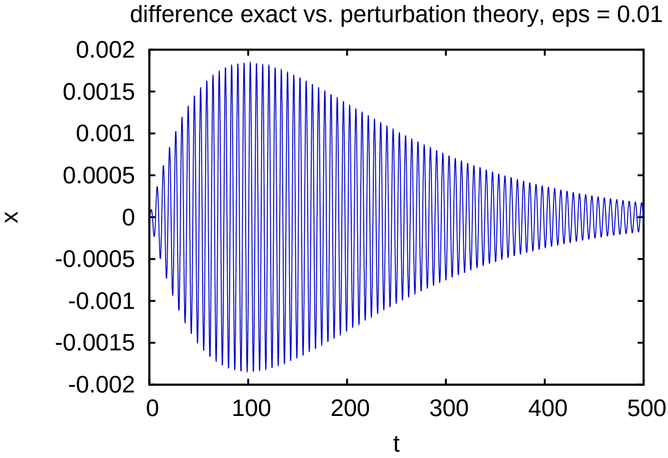
$$\begin{aligned} \frac{\partial A}{\partial t_1} + A &= 0 \\ \frac{\partial B}{\partial t_1} + B &= 0 \end{aligned}$$

or

$$\begin{aligned} A &= \hat{A} \exp\left(-t_1\right) \\ B &= \hat{B} \exp\left(-t_1\right) \end{aligned}$$

insert this into ε^0 solution, initial conditions:

$$x_0 = \exp\left(-\varepsilon t\right) \cos t$$


difference exact vs. perturbation theory, eps = 0.01
0.002
0.0015
0.001
0.0005
0
-0.0005
-0.001
-0.0015
-0.002
x
0
100
200
300
400
500
t

Chapman–Enskog expansion

- original LBE:

$$n_i(\vec{r} + \vec{c}_i h, t + h) - n_i(\vec{r}, t) = \Delta_i$$

- desired: continuum limit $h \rightarrow 0$, $\vec{c}_i$ fixed
- set $h = \varepsilon h_0$
- expansion parameter $\varepsilon \ll 1$, $\varepsilon \rightarrow 0$
- write $t_1 = t$, $\vec{r}_1 = \vec{r}$
- yields:

$$n_i(\vec{r}_1 + \vec{c}_i \varepsilon h_0, t_1 + \varepsilon h_0) - n_i(\vec{r}_1, t_1) = \Delta_i$$

- *two* time scales:
  - waves: time $\sim$ length
  - diffusion: time $\sim$ (length)^2
- second time scale: $t_2 = \varepsilon t$

- study LBE:

$$n_i(\vec{r}_1 + \varepsilon \vec{c}_i h_0, t_1 + \varepsilon h_0, t_2 + \varepsilon^2 h_0) - n_i(\vec{r}_1, t_1, t_2) = \Delta_i$$

- $$\frac{\partial}{\partial \vec{r}} = \frac{\partial}{\partial \vec{r}_1}$$

- $$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2}$$

- set

$$D_i = \varepsilon h_0 c_{i\alpha} \partial_{1\alpha} + \varepsilon h_0 \partial_{t1} + \varepsilon^2 h_0 \partial_{t2}$$

$$D_i n_i = \left[1 - \frac{D_i}{2} + \frac{D_i^2}{12} + \dots \right] \Delta_i$$

ε orders

- ε^1:

$$\left(h_0 c_{i\alpha} \partial_{1\alpha} + h_0 \partial_{t1}\right) n_i^{(0)} = \Delta_i^{(1)}$$

- ε^2:

$$\begin{aligned} & h_0 \partial_{t2} n_i^{(0)} + \left(h_0 c_{i\alpha} \partial_{1\alpha} + h_0 \partial_{t1}\right) n_i^{(1)} \\ = \quad & \Delta_i^{(2)} - \frac{1}{2} \left(h_0 c_{i\alpha} \partial_{1\alpha} + h_0 \partial_{t1}\right) \Delta_i^{(1)} \end{aligned}$$

or

$$h_0 \partial_{t2} n_i^{(0)} + \frac{1}{2} \left(h_0 c_{i\alpha} \partial_{1\alpha} + h_0 \partial_{t1}\right) \left(n_i^{(1)\star} + n_i^{(1)}\right) = \Delta_i^{(2)}$$

Zeroth velocity moment: Mass conservation

$$\partial_{t_1} \rho + \partial_{1\alpha} j_\alpha = 0$$

$$\partial_{t_2} \rho = 0$$

Hence,

👍 continuity equation OK!!!

First velocity moment: Momentum conservation

$$\partial_{t_1} j_\alpha + \partial_{1\beta} \Pi^{(0)}_{\alpha\beta} = 0$$

$$\partial_{t_2} j_\alpha + \frac{1}{2} \partial_{1\beta} \left(\Pi^{\star(1)}_{\alpha\beta} + \Pi^{(1)}_{\alpha\beta} \right) = 0$$

comparison with Navier–Stokes:

Euler stress:

$$\Pi^{(0)}_{\alpha\beta} = \rho c_s^2 \delta_{\alpha\beta} + \rho u_\alpha u_\beta$$

Newtonian viscous stress:

$$\frac{\varepsilon}{2} \left(\Pi^{\star(1)}_{\alpha\beta} + \Pi^{(1)}_{\alpha\beta} \right) = -\sigma_{\alpha\beta}$$

Second velocity moment: A useful relation

$$\partial_{t_1} \Pi^{(0)}_{\alpha\beta} + \partial_{1\gamma} \Phi^{(0)}_{\alpha\beta\gamma} = h_0^{-1} \left(\Pi^{\star(1)}_{\alpha\beta} - \Pi^{(1)}_{\alpha\beta} \right)$$

from explicit form of n_i^{eq}:

$$\Phi^{(0)}_{\alpha\beta\gamma} = \rho c_s^2 \left(u_\alpha \delta_{\beta\gamma} + u_\beta \delta_{\alpha\gamma} + u_\gamma \delta_{\alpha\beta} \right)$$

use continuity and Euler for

$$\partial_{t_1} \Pi^{(0)}_{\alpha\beta} = \partial_{t_1} (\rho c_s^2 \delta_{\alpha\beta} + \rho u_\alpha u_\beta) = \ldots$$

$\Rightarrow$ (neglecting terms $O(u^3)$):

$$\Pi^{\star(1)}_{\alpha\beta} - \Pi^{(1)}_{\alpha\beta} = h_0 \rho c_s^2 \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha \right)$$

(details see next three slides)

Calculation of Φ

$$n_i^{eq}(\rho, \vec{u}) = w_i \rho \left(1 + \frac{\vec{u} \cdot \vec{c}_i}{c_s^2} + \frac{(\vec{u} \cdot \vec{c}_i)^2}{2 c_s^4} - \frac{u^2}{2 c_s^2} \right)$$

hence

$$\begin{aligned} \Phi_{\alpha\beta\gamma} &= \frac{\rho}{c_s^2} u_\delta \sum_i w_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta} \\ &= \frac{\rho}{c_s^2} u_\delta c_s^4 \left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} \right) \\ &= \rho c_s^2 \left(\delta_{\alpha\beta} u_\gamma + \delta_{\alpha\gamma} u_\beta + \delta_{\beta\gamma} u_\alpha \right) \end{aligned}$$

Equation of motion for the Euler stress

pure Euler hydrodynamics

$$\Pi_{\alpha\beta} = \rho c_s^2 \delta_{\alpha\beta} + \rho u_\alpha u_\beta$$

Euler equations:

$$\begin{aligned}
\frac{D}{Dt} &= \frac{\partial}{\partial t} + u_\gamma \partial_\gamma \\
\frac{D}{Dt}\rho &= -\rho \partial_\gamma u_\gamma \\
\rho \frac{D}{Dt} u_\alpha &= -c_s^2 \partial_\alpha \rho \\
\frac{D}{Dt}\Pi_{\alpha\beta} &= \left(c_s^2 \delta_{\alpha\beta} + u_\alpha u_\beta\right) \frac{D}{Dt}\rho + u_\alpha \rho \frac{D}{Dt} u_\beta + u_\beta \rho \frac{D}{Dt} u_\alpha \\
&= -\rho \left(c_s^2 \delta_{\alpha\beta} + u_\alpha u_\beta\right) \partial_\gamma u_\gamma - c_s^2 u_\alpha \partial_\beta \rho - c_s^2 u_\beta \partial_\alpha \rho
\end{aligned}$$

neglect $O(u^3)$:

$$\frac{\partial}{\partial t}\Pi_{\alpha\beta} + u_\gamma c_s^2 \delta_{\alpha\beta} \partial_\gamma \rho = -c_s^2 \delta_{\alpha\beta} \rho \partial_\gamma u_\gamma - c_s^2 u_\alpha \partial_\beta \rho - c_s^2 u_\beta \partial_\alpha \rho$$

$$
\begin{aligned}
& \frac{\partial}{\partial t}\Pi_{\alpha\beta} + c_s^2 \delta_{\alpha\beta}\partial_\gamma\left(\rho u_\gamma\right) + c_s^2 u_\alpha \partial_\beta \rho + c_s^2 u_\beta \partial_\alpha \rho = 0 \\
& \frac{\partial}{\partial t}\Pi_{\alpha\beta} + c_s^2 \delta_{\alpha\beta}\partial_\gamma\left(\rho u_\gamma\right) + c_s^2 \partial_\beta\left(\rho u_\alpha\right) + c_s^2 \partial_\alpha\left(\rho u_\beta\right) \\
= \quad & c_s^2 \rho \partial_\beta u_\alpha + c_s^2 \rho \partial_\alpha u_\beta \\
& \frac{\partial}{\partial t}\Pi_{\alpha\beta} + \partial_\gamma\left\{\rho c_s^2\left(\delta_{\alpha\beta} u_\gamma + \delta_{\beta\gamma} u_\alpha + \delta_{\alpha\gamma} u_\beta\right)\right\} \\
= \quad & c_s^2 \rho \partial_\beta u_\alpha + c_s^2 \rho \partial_\alpha u_\beta \\
& \frac{\partial}{\partial t}\Pi_{\alpha\beta} + \partial_\gamma \Phi_{\alpha\beta\gamma} = \rho c_s^2\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right)
\end{aligned}
$$

Linear collision operator

$$\begin{aligned}\Delta_i &= \Delta_i^{(0)} + \varepsilon \Delta_i^{(1)} + O(\varepsilon^2) \\ &= \varepsilon \Delta_i^{(1)} + O(\varepsilon^2)\end{aligned}$$

$O(\varepsilon^2)$ does not contribute to hydrodynamics $\Rightarrow$ ignore

$$\Delta_i^{(1)} = \sum_j \left. \frac{\partial \Delta_i}{\partial n_j} \right|_{\{n_k^0\}} n_j^{(1)} = \sum_j L_{ij} n_j^{(1)}$$

i. e.

$$\Delta_i = \sum_j L_{ij} \left(n_j - n_j^{eq} \right)$$

The linear collision process

$$\begin{aligned} n_i^{neq} &:= n_i - n_i^{eq} \\ n_i^{\star} &= n_i + \sum_j L_{ij} n_j^{neq} \\ n_i^{neq\star} &= n_i^{neq} + \sum_j L_{ij} n_j^{neq} \\ \Gamma_{ij} &:= \delta_{ij} + L_{ij} \\ n_i^{neq\star} &= \sum_j \Gamma_{ij} n_j^{neq} \end{aligned}$$

$\Gamma =$???

simplest choice: Lattice BGK:

$$\Gamma_{ij} = \left(1 - \frac{1}{\tau} \right) \delta_{ij}$$

study here the MRT (multi relaxation time) framework!

$$n_i^{neq\star} = \sum_j \Gamma_{ij} n_j^{neq}$$

$\sum_i \Rightarrow$

$$\begin{aligned} 0 &= \sum_j \left(\sum_i \Gamma_{ij} \right) n_j^{neq} \\ 0 &= \sum_i \Gamma_{ij} \\ e_{0i} &:= 1 \\ e_{0j} \cdot 0 &= \sum_i e_{0i} \Gamma_{ij} \end{aligned}$$

i. e. $\vec{e}_0$ is left eigenvector, eigenvalue zero.

$$n_i^{neq\star} = \sum_j \Gamma_{ij} n_j^{neq}$$

$\sum_i c_{ix} \Rightarrow$

$$\begin{aligned} 0 &= \sum_j \left(\sum_i c_{ix} \Gamma_{ij} \right) n_j^{neq} \\ 0 &= \sum_i c_{ix} \Gamma_{ij} \\ e_{1i} &:= c_{ix} \\ e_{1j} \cdot 0 &= \sum_i e_{1i} \Gamma_{ij} \end{aligned}$$

i. e. $\vec{e}_1$ is left eigenvector, eigenvalue zero.
analogous: $e_{2i} = c_{iy}$, $e_{3i} = c_{iz}$

$$n_i^{neq\star} = \sum_j \Gamma_{ij} n_j^{neq}$$

$\sum_i c_{i\gamma} c_{i\gamma}$ (bulk stress) $\Rightarrow$ (bulk stress relaxation with γ_b):

$$\begin{aligned}
\Pi_{\gamma\gamma}^{neq\star} &= \sum_j \left(\sum_i c_{i\gamma} c_{i\gamma} \Gamma_{ij} \right) n_j^{neq} \\
\Pi_{\gamma\gamma}^{neq\star} &= \gamma_b \Pi_{\gamma\gamma}^{neq} = \gamma_b \sum_j n_j^{neq} c_{j\gamma} c_{j\gamma} = \sum_j \left(\gamma_b c_{j\gamma} c_{j\gamma} \right) n_j^{neq} \\
\gamma_b c_{j\gamma} c_{j\gamma} &= \sum_i c_{i\gamma} c_{i\gamma} \Gamma_{ij} \\
e_{4i} &:= c_{i\gamma} c_{i\gamma} \\
e_{4j} \gamma_b &= \sum_i e_{4i} \Gamma_{ij}
\end{aligned}$$

i. e. $\vec{e}_4$ is left eigenvector, eigenvalue γ_b

…and so on! $\vec{e}_5, \ldots, \vec{e}_9$: five shear stresses, eigenvalue γ_s (same value for symmetry reasons)
$\vec{e}_{10}, \ldots, \vec{e}_{18}$ 9 “kinetic modes”, “ghost modes” (higher–order polynomials in the $\vec{c}_i$)
i. e. we do not know Γ directly, but its eigenvalues and eigenvectors!
generally (eigenvalues γ_i)

$$e_{kj}\gamma_k = \sum_i e_{ki}\Gamma_{ij}$$

$$|\gamma_i| \leq 1$$

for linear stability!

Modes

set

$$
\begin{aligned}
m_k &= \sum_j e_{kj} n_j \\
m_k^{neq\star} &= \sum_i e_{ki} n_i^{neq\star} = \sum_i e_{ki} \sum_j \Gamma_{ij} n_j^{neq} \\
&= \sum_j \left(\sum_i e_{ki} \Gamma_{ij} \right) n_j^{neq} = \sum_j e_{kj} \gamma_k n_j^{neq} = \gamma_k m_k^{neq}
\end{aligned}
$$

I. e. the relaxation process is simple in mode space!

- $\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = 0$ (mass and momentum conservation)
- $\gamma_4 = \gamma_b$ (bulk stress)
- $\gamma_5 = \ldots = \gamma_9 = \gamma_s$ (shear stress)
- $\gamma_{10} = \ldots = \gamma_{18} = 0$ (simplest choice, not necessary)

Orthogonality

scalar product:

$$\langle \vec{n}' | \vec{n} \rangle = \sum_i w_i n_i' n_i$$

Claim: It is possible to pick the eigenvectors in such a way that they satisfy

$$\langle \vec{e}_k | \vec{e}_l \rangle = \mathcal{N}_k \delta_{kl}$$

where the $\mathcal{N}_k$ are just normalization constants.
Proof: Either you understand group theory pretty well, or you do an explicit Gram–Schmidt orthogonalization! Result is tabulated in the review!

$$\delta_{kl} = \frac{1}{\mathcal{N}_k} \sum_i w_i e_{ki} e_{li}$$

$$= \sum_i \sqrt{\frac{w_i}{\mathcal{N}_k}} e_{ki} \sqrt{\frac{w_i}{\mathcal{N}_l}} e_{li}$$

$$\hat{e}_{ki} := \sqrt{\frac{w_i}{\mathcal{N}_k}} e_{ki}$$

$$\delta_{kl} = \sum_i \hat{e}_{ki} \hat{e}_{li}$$

i. e. $\hat{e}_{ki}$ is a standard orthogonal matrix with Euclidean scalar product!

$$
\begin{aligned}
m_k &= \sum_i e_{ki} n_i = \sum_i \sqrt{\frac{\mathcal{N}_k}{w_i}} \hat{e}_{ki} n_i \\
\hat{m}_k &:= \frac{1}{\sqrt{\mathcal{N}_k}} m_k \\
\hat{n}_i &:= \frac{1}{\sqrt{w_i}} n_i \\
\hat{m}_k &= \sum_i \hat{e}_{ki} \hat{n}_i
\end{aligned}
$$

orthonormal transformation, trivial to invert!

Linear stress relaxation and viscosities

$$\begin{aligned}
\Pi_{\alpha\beta} &= \bar{\Pi}_{\alpha\beta} + \frac{1}{3}\delta_{\alpha\beta}\Pi_{\gamma\gamma} \\
\bar{\Pi}^{\star neq}_{\alpha\beta} &= \gamma_s \bar{\Pi}^{neq}_{\alpha\beta} \\
\Pi^{\star neq}_{\gamma\gamma} &= \gamma_b \Pi^{neq}_{\gamma\gamma} \\
\Pi^{\star neq}_{\alpha\beta} - \Pi^{neq}_{\alpha\beta} &= \bar{\Pi}^{\star neq}_{\alpha\beta} - \bar{\Pi}^{neq}_{\alpha\beta} + \frac{1}{3}\delta_{\alpha\beta}\left(\Pi^{\star neq}_{\gamma\gamma} - \Pi^{neq}_{\gamma\gamma}\right) \\
&= \left(\gamma_s - 1\right)\bar{\Pi}^{neq}_{\alpha\beta} + \frac{1}{3}\delta_{\alpha\beta}\left(\gamma_b - 1\right)\Pi^{neq}_{\gamma\gamma}
\end{aligned}$$

on the other hand, we had derived

$$\begin{aligned}\Pi^{\star neq}_{\alpha\beta} - \Pi^{neq}_{\alpha\beta} &= h\rho c_s^2 \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right) \\ &= h\rho c_s^2 \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\delta_{\alpha\beta}\partial_\gamma u_\gamma\right) \\ &+ \frac{2}{3} h\rho c_s^2 \delta_{\alpha\beta}\partial_\gamma u_\gamma\end{aligned}$$

comparison:

$$\begin{aligned}\left(\gamma_s - 1\right)\bar{\Pi}^{neq}_{\alpha\beta} &= h\rho c_s^2 \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\delta_{\alpha\beta}\partial_\gamma u_\gamma\right) \\ \left(\gamma_b - 1\right)\Pi^{neq}_{\gamma\gamma} &= 2h\rho c_s^2 \partial_\gamma u_\gamma\end{aligned}$$

or

$$\begin{aligned}\bar{\Pi}^{neq}_{\alpha\beta} &= \frac{h\rho c_s^2}{\gamma_s - 1}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\delta_{\alpha\beta}\partial_\gamma u_\gamma\right) \\ \Pi^{neq}_{\gamma\gamma} &= \frac{2h\rho c_s^2}{\gamma_b - 1}\partial_\gamma u_\gamma\end{aligned}$$

Chapman–Enskog told us: Newtonian viscous stress is

$$-\sigma_{\alpha\beta} = \frac{1}{2}\left(\Pi^{neq\star}_{\alpha\beta} + \Pi^{neq}_{\alpha\beta}\right)$$

hence

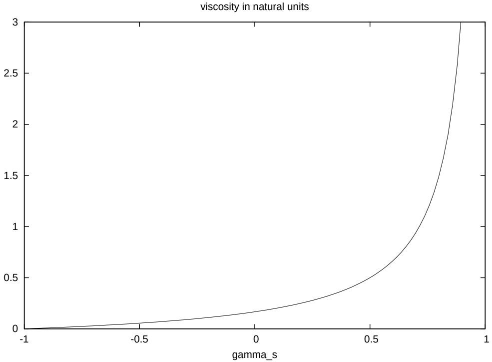
$$\begin{aligned}
&\frac{1}{2}\left(\bar{\Pi}^{neq\star}_{\alpha\beta} + \bar{\Pi}^{neq}_{\alpha\beta}\right) = \frac{1}{2}\left(\gamma_s + 1\right)\bar{\Pi}^{neq}_{\alpha\beta} \\
&= \frac{h\rho c_s^2}{2}\frac{\gamma_s + 1}{\gamma_s - 1}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\delta_{\alpha\beta}\partial_\gamma u_\gamma\right) \\
&= -\eta\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\delta_{\alpha\beta}\partial_\gamma u_\gamma\right)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{1}{3}\delta_{\alpha\beta}\left(\Pi^{neq\star}_{\gamma\gamma} + \Pi^{neq}_{\gamma\gamma}\right) = \frac{1}{2}\frac{1}{3}\delta_{\alpha\beta}\left(\gamma_b + 1\right)\Pi^{neq}_{\gamma\gamma} \\
&= \frac{h\rho c_s^2}{3}\frac{\gamma_b + 1}{\gamma_b - 1}\delta_{\alpha\beta}\partial_\gamma u_\gamma \\
&= -\zeta\delta_{\alpha\beta}\partial_\gamma u_\gamma
\end{aligned}$$

read off viscosities:

$$\eta = \frac{h \rho c_s^2}{2} \frac{1 + \gamma_s}{1 - \gamma_s} \qquad \zeta = \frac{h \rho c_s^2}{3} \frac{1 + \gamma_b}{1 - \gamma_b}$$



- $|\gamma_i| \leq 1 \Leftrightarrow$ positive viscosities!
- *any* viscosity values are accessible!