$\{q_i\}, \{p_i\} \longrightarrow \{Q_i\}, \{P_i\}$ canonical

Generating Functions for Canonical Transformations

Principle of least action for old variables:

$$0 = \delta \int_{t_1}^{t_2} dt \left\{ \sum_i p_i \dot{q}_i - H(\{p_i\}, \{q_i\}, t) \right\}, \quad \begin{array}{l} q_i(t_1) \text{ fixed} \\ q_i(t_2) \text{ fixed} \end{array}$$

for the new ones:

$$0 = \delta \int_{t_1}^{t_2} dt \left\{ \sum_i P_i \dot{Q}_i - H'(\{P_i\}, \{Q_i\}, t) \right\} \quad \begin{array}{l} Q_i(t_1) \text{ fixed} \\ Q_i(t_2) \text{ fixed} \end{array}$$

these must be equivalent $\longrightarrow$

$$\sum_i p_i \dot{q}_i - H(\{p_i\},\{q_i\},t) = \sum_i P_i \dot{Q}_i - H'(\{P_i\},\{Q_i\},t) + \frac{d}{dt} F(\{q_i\},\{Q_i\},t)$$

$$dF = \sum_i p_i \, dq_i - \sum_i P_i \, dQ_i + (H' - H)\, dt$$

$$\boxed{p_i = \frac{\partial F}{\partial q_i} \qquad \frac{\partial F}{\partial t} = H' - H \qquad -P_i = \frac{\partial F}{\partial Q_i}}$$

F is called generating function

Example: $F = \sum_i q_i Q_i$

$$p_i = \frac{\partial F}{\partial q_i} = Q_i \qquad P_i = -\frac{\partial F}{\partial Q_i} = -q_i$$

just exchange positions & momenta!

Other generating functions via Legendre transforms

example: $G := F + \sum_i P_i Q_i$

$$dG = dF + \sum_i P_i \, dQ_i + \sum_i Q_i \, dP_i =$$

$$= \sum_i p_i \, dq_i - \cancel{\sum_i P_i \, dQ_i} + (H' - H)\, dt + \cancel{\sum_i P_i \, dQ_i}$$

$$+ \sum_i Q_i \, dP_i =$$

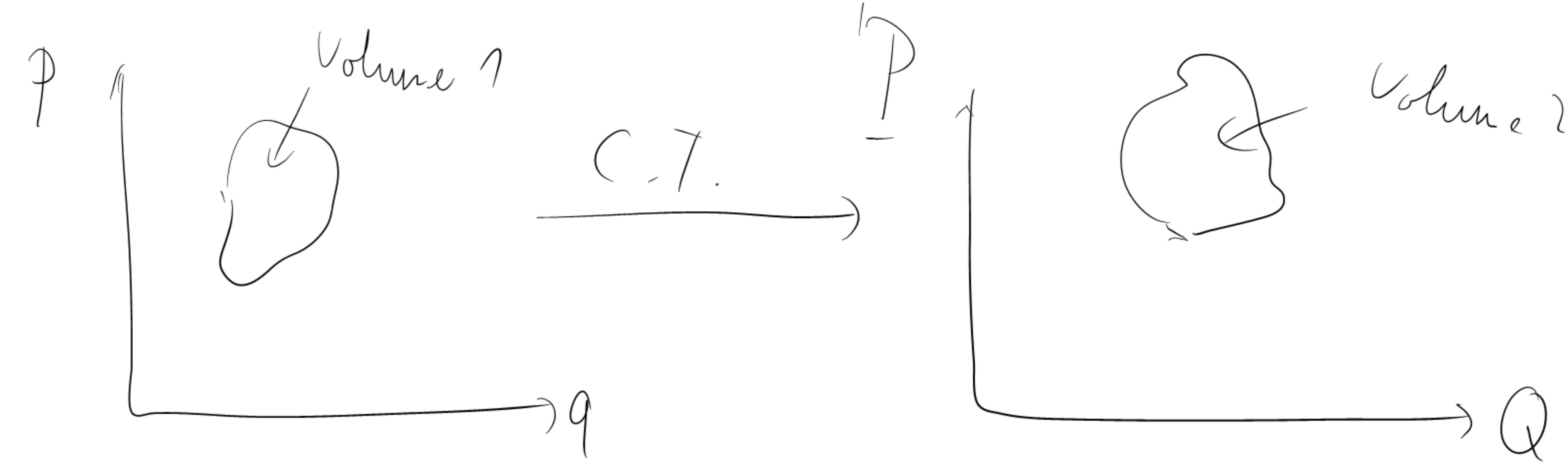
$$dG = \sum_i p_i \, dq_i + \sum_i Q_i \, dP_i + (H' - H) \, dt$$

$$p_i = \frac{\partial G}{\partial q_i} \qquad Q_i = \frac{\partial G}{\partial P_i} \qquad H' = H + \frac{\partial G}{\partial t}$$

<u>Example:</u> $G = \sum_i q_i P_i \Rightarrow p_i = P_i, \; Q_i = q_i$

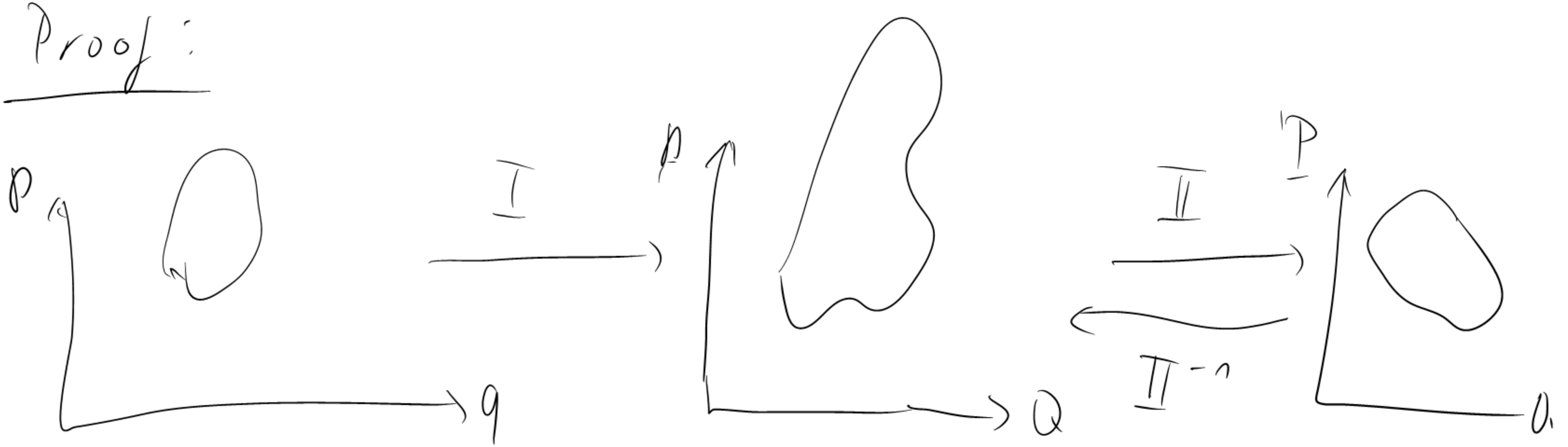
generates the <u>identical</u> transformation!

<u>Phase Space Volume is INVARIANT under Canonical Transformations</u>



Claim: Volume 1 = Volume 2

Proof:



Distortion factor for the volume:

Determinant of the Jacobian:

$$\mathrm{I}: \quad \det\left(\frac{\partial Q_i}{\partial q_j}\right)$$

$$\mathrm{II}^{-1}: \quad \det\left(\frac{\partial p_i}{\partial P_j}\right)$$

claim:

$$\det\left(\frac{\partial Q_i}{\partial q_j}\right) = \det\left(\frac{\partial p_i}{\partial P_j}\right)$$

Use generating function $G(\{q_i\}, \{P_i\})$:

$$Q_i = \frac{\partial G}{\partial P_i} \Rightarrow \frac{\partial Q_i}{\partial q_j} = \frac{\partial^2 G}{\partial P_i \partial q_j}$$

$$p_i = \frac{\partial G}{\partial q_i} \Rightarrow \frac{\partial p_i}{\partial P_j} = \frac{\partial^2 G}{\partial P_j \partial q_i}$$

$\rightarrow$ matrices are transposed wrt each other

$\rightarrow$ determinants are the same! $\square$

Infinitesimal Canonical Transformations

$$G(\{q_i\},\{P_i\}) = \sum_i q_i P_i + \varepsilon\, \sigma(\{q_i\},\{P_i\}) + O(\varepsilon^2)$$

ε small, $\varepsilon \to 0$

$$p_i = \frac{\partial G}{\partial q_i} = P_i + \varepsilon \frac{\partial \sigma}{\partial q_i} + O(\varepsilon^2) \simeq$$

$$\simeq P_i + \varepsilon \frac{\partial}{\partial q_i} \sigma(\{q_i\},\{p_i\}) + O(\varepsilon^2)$$

$$Q_i = \frac{\partial G}{\partial P_i} = q_i + \varepsilon \frac{\partial \sigma}{\partial P_i} + O(\varepsilon^2) \simeq$$

$$\simeq q_i + \varepsilon \frac{\partial}{\partial p_i} \sigma(\{q_i\},\{p_i\}) + O(\varepsilon^2)$$

$$P_i = p_i - \varepsilon \frac{\partial}{\partial q_i} \sigma(\{q_i\}, \{p_i\}) + O(\varepsilon^2)$$

$$Q_i = q_i + \varepsilon \frac{\partial}{\partial p_i} \sigma(\{q_i\}, \{p_i\}) + O(\varepsilon^2)$$

Consider now the actual <u>time</u> development

$$q_i(t) \equiv q_i \longrightarrow Q_i \equiv q_i(t+\tau) \qquad \tau \to 0$$

$$p_i(t) \equiv p_i \longrightarrow P_i \equiv p_i(t+\tau) \qquad \tau \to 0$$

$$Q_i = q_i + \tau \dot{q}_i + O(\tau^2) = q_i + \tau \frac{\partial H}{\partial p_i} + O(\tau^2)$$

$\rightarrow$ Set $G = H$ $\quad \varepsilon = \tau$

Study P_i:

$$P_i = p_i + \tau \dot{p}_i + O(\tau^2) = p_i - \tau \frac{\partial H}{\partial q_i} + O(\tau^2)$$

$\rightarrow$ works for P_i too!

$\Rightarrow$ Hamiltonian dynamics is in itself just a canonical transformation!

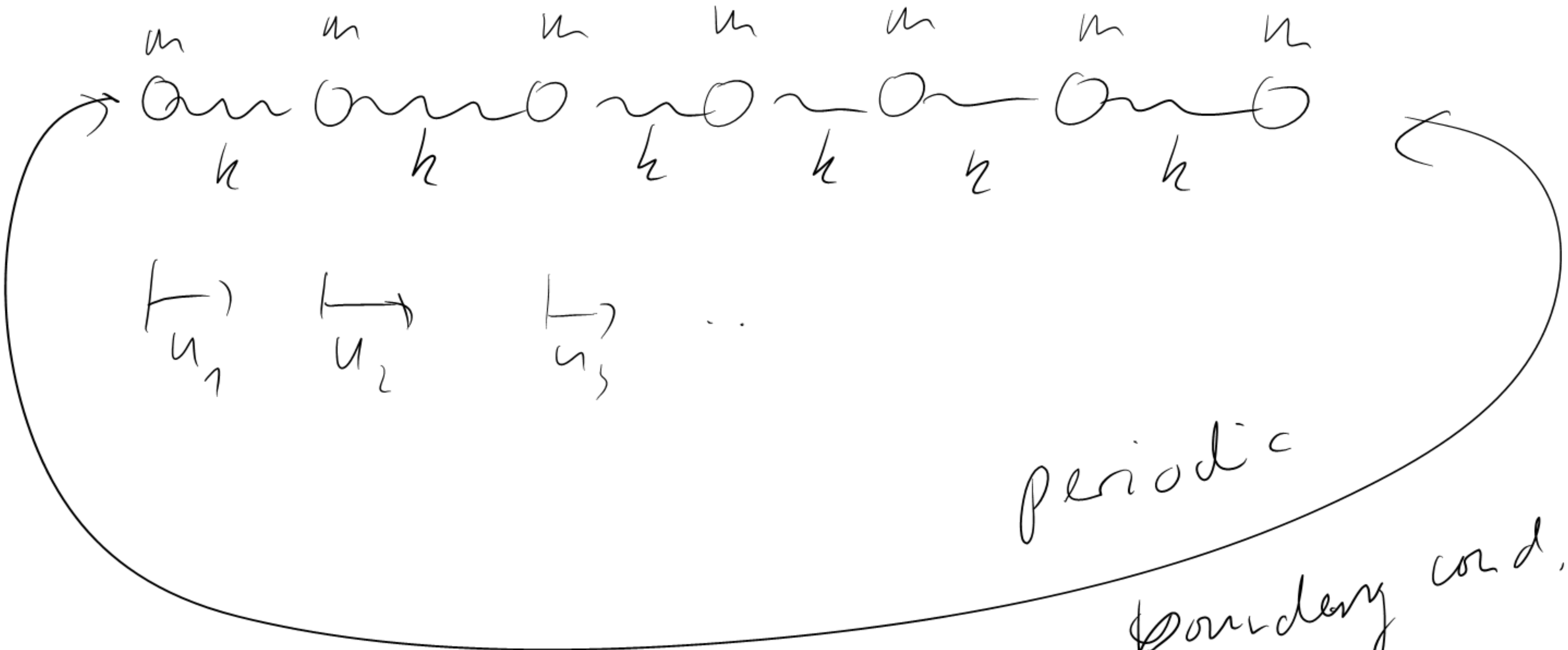
Consequence:

LIOUVILLE'S THEOREM:

Phase space volume is invariant under Hamiltonian dynamics!!!

FIELD THEORY

<u>Example</u>: Linear harmonic chain N mass points



periodic boundary cond.

$u_{N+1} = u_1$

$$L = \frac{m}{2}\left(\dot{u}_1^2 + \dot{u}_2^2 + \dots + \dot{u}_N^2\right) - \frac{k}{2}\Big[(u_2-u_1)^2 + (u_3-u_2)^2 +$$

$$+ \dots + (u_N - u_{N-1})^2 + (\underbrace{u_{N+1}}_{=u_1} - u_N)^2\Big]$$

$$\frac{\partial L}{\partial \dot{u}_1} = m\dot{u}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_1}\right) = m\ddot{u}_1$$

$$\frac{\partial L}{\partial u_1} = -k(u_1 - u_2) - k(u_1 - u_N) = -2k\,u_1 + k u_2 + k u_N$$

$$u_i \longrightarrow u(x) \qquad \sum_{i=1}^{N} \longrightarrow \frac{1}{a}\int_0^{Na} dx \qquad Na = R$$

a: lattice spacing

$$u_{i+1} - u_i = a \frac{\partial}{\partial x} u$$

Lagrangian in the continuum limit:

$$L = \frac{1}{a} \int_0^R dx \left\{ \frac{m}{2} \dot{u}(x,t)^2 - \frac{k}{2} \left[\frac{\partial}{\partial x} u(x,t) \right]^2 a^2 \right\}$$

$$= \int_0^R dx \; \mathcal{L}(x,t) \qquad \text{Lagrange density}$$

$$\ell = \mathcal{L}\left(u, \dot{u}, \frac{\partial}{\partial x} u\right) \qquad \partial_t \equiv \frac{\partial}{\partial t}$$

cf. $L(q, \dot{q})$

$$S = \int dt \int d^3x\, \mathcal{L}\left(\varphi(\vec{r}, t), \frac{\partial}{\partial t}\varphi, \vec{\nabla}\varphi\right)$$

$$\delta S = \int dt \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} \cdot \delta(\vec{\nabla}\varphi) + \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)}\delta(\partial_t \varphi) \right\}$$

$$= \int dt \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} - \vec{\nabla}\cdot\frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \right\} \delta\varphi$$

boundary terms vanish by suitable formulation.

Euler-Lagrange eq.:

$$\boxed{\vec{\nabla}\cdot\frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} + \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi}}$$

$$\mathcal{L} = \frac{m}{2a}\dot{u}^2 - \frac{ka}{2}\left(\frac{\partial}{\partial x}u\right)^2 \Rightarrow \frac{\partial \mathcal{L}}{\partial u} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_x u)} = -ka(\partial_x u) \Rightarrow \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x u)} = -ka\,\partial_x^2 u$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_t u)} = \frac{m}{a}(\partial_t u) \Rightarrow \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t u)} = \frac{m}{a}\partial_t^2 u$$

$$\frac{m}{a}\partial_t^2 u - ka\,\partial_x^2 u = 0$$

$$\left(\frac{m}{ka^2}\frac{\partial}{\partial t^2} - \frac{\partial}{\partial x^2}\right)u = 0$$

wave equation

$$\frac{1}{c^2} = \frac{m}{ka^2} \qquad \boxed{c^2 = \frac{ka^2}{m}}$$

$\downarrow$

c: speed of sound