Theory of Critical Phenomena

We had: $$G(r) = \langle S(r)\, S(0)\rangle - \langle S\rangle^2$$

$$\sim \frac{1}{r^{d-2+\eta}}\, \tilde{G}\left(\frac{r}{\xi}\right)$$

corr. length

$$t = \frac{T - T_c}{T_c} \qquad \xi \propto |t|^{-\nu}$$

"independent" exponents: ν, η

$$C_v \sim |t|^{-\alpha} \qquad \alpha = 2 - d\nu$$

$$\chi \sim |t|^{-\gamma} \qquad \gamma = \nu(2-\eta)$$

$$m \sim |t|^{\beta} \qquad \beta = \frac{\nu}{2}(d-2+\eta)$$

$$\alpha + 2\beta + \gamma = 2 \qquad \underbrace{\alpha = 2 - d\nu}$$

$$2\beta + \gamma = d\nu \qquad \underbrace{\gamma = d\nu - 2\beta}$$

Upper Critical Dimension / Ginzburg Criterion

When is Mean Field Theory OK?

Requirement: Fluctuations $\ll$ Mean!

$$\chi = \frac{L^d}{k_B T}\left[\langle m^2\rangle - \langle m\rangle^2\right]$$

fluctuations within a correlation volume

$$\langle \delta m^2 \rangle = k_B T \chi \xi^{-d} \propto |t|^{-\gamma} |t|^{+d\nu}$$

$$\langle m\rangle \sim |t|^{\beta} \qquad \langle m\rangle^2 \sim |t|^{2\beta}$$

$$\frac{\langle \delta m^2\rangle}{\langle m\rangle^2} \propto |t|^{d\nu - \gamma - 2\beta}$$

if this is $\ll 1$, then Mean Field OK

criterion for MF OK: $\boxed{d\nu - \gamma - 2\beta > 0}$

Mean Field exponents: $\nu = 1/2$, $\gamma = 1$, $\beta = 1/2$

$d \cdot 1/2 - 1 - 1 > 0 \Rightarrow \underline{\underline{d > 4}}$ upper critical dimension

critical scaling:

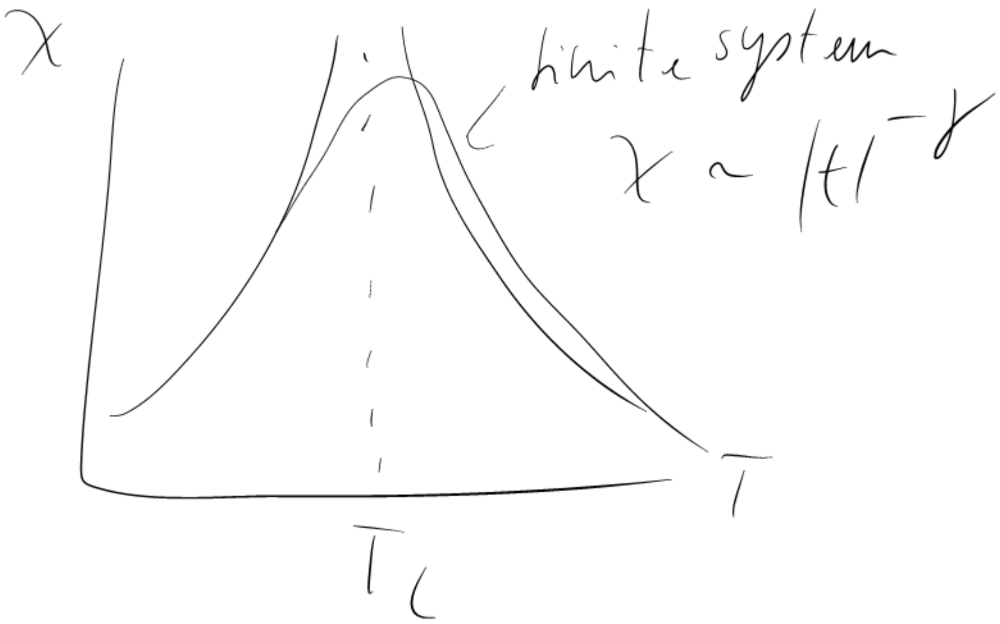
$$\text{exponent} = d\nu - \gamma - 2\beta = 2 - \underbrace{(\alpha + \gamma + 2\beta)}_{=2} = 0$$

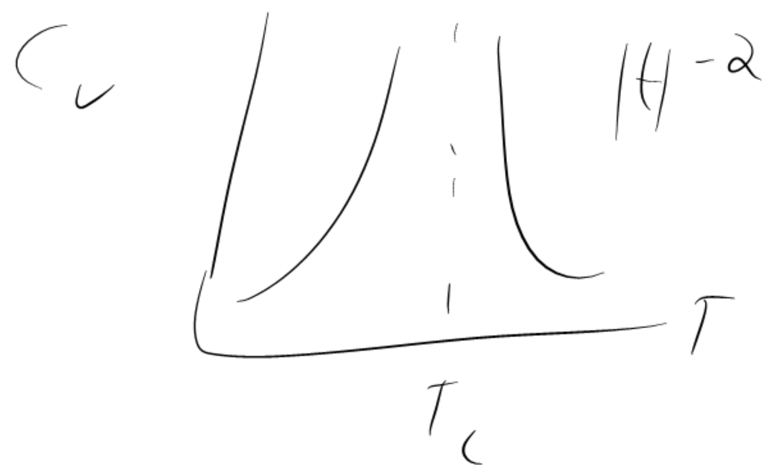
fluctuations & mean are of the same order!

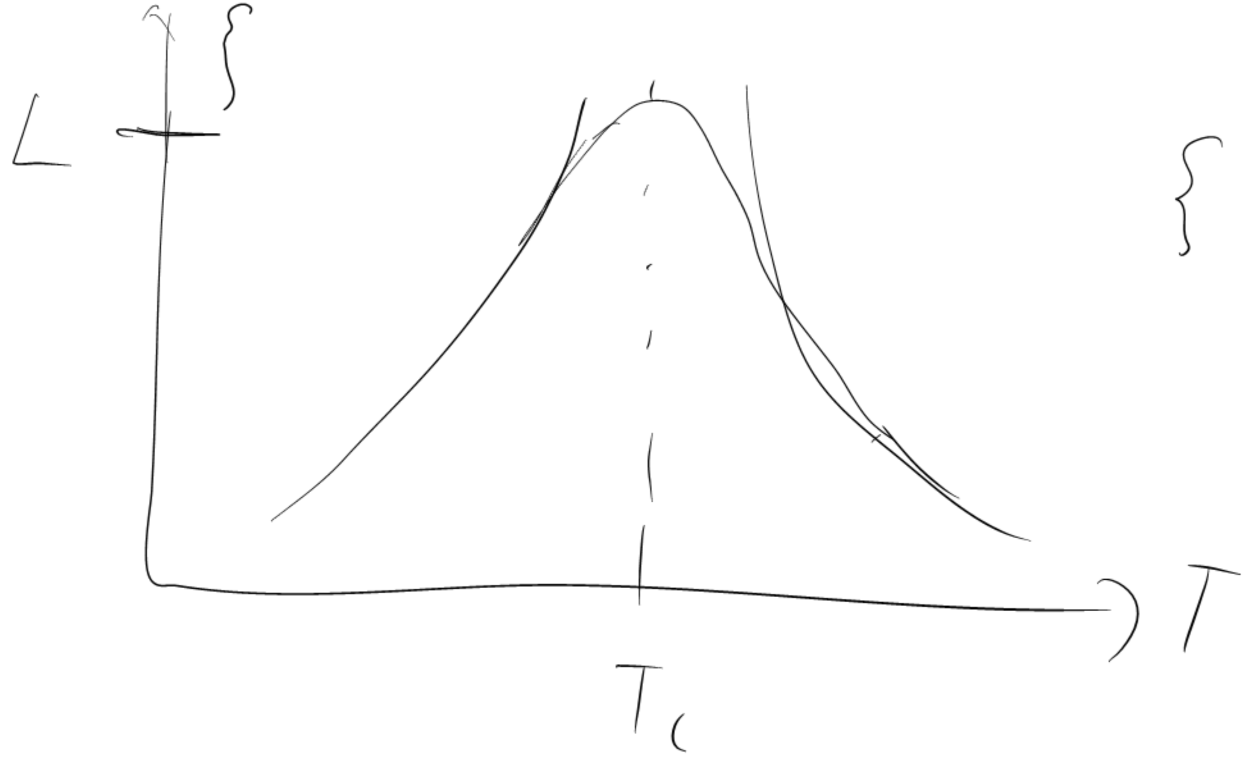
Phase Transitions in Finite Systems

$$Z = \sum_{\{S_i\}} e^{-\beta \mathcal{H}} \qquad F = -\frac{1}{\beta} \ln Z$$

FINITE SUM $\rightarrow$ ANALYTIC wrt. T !!





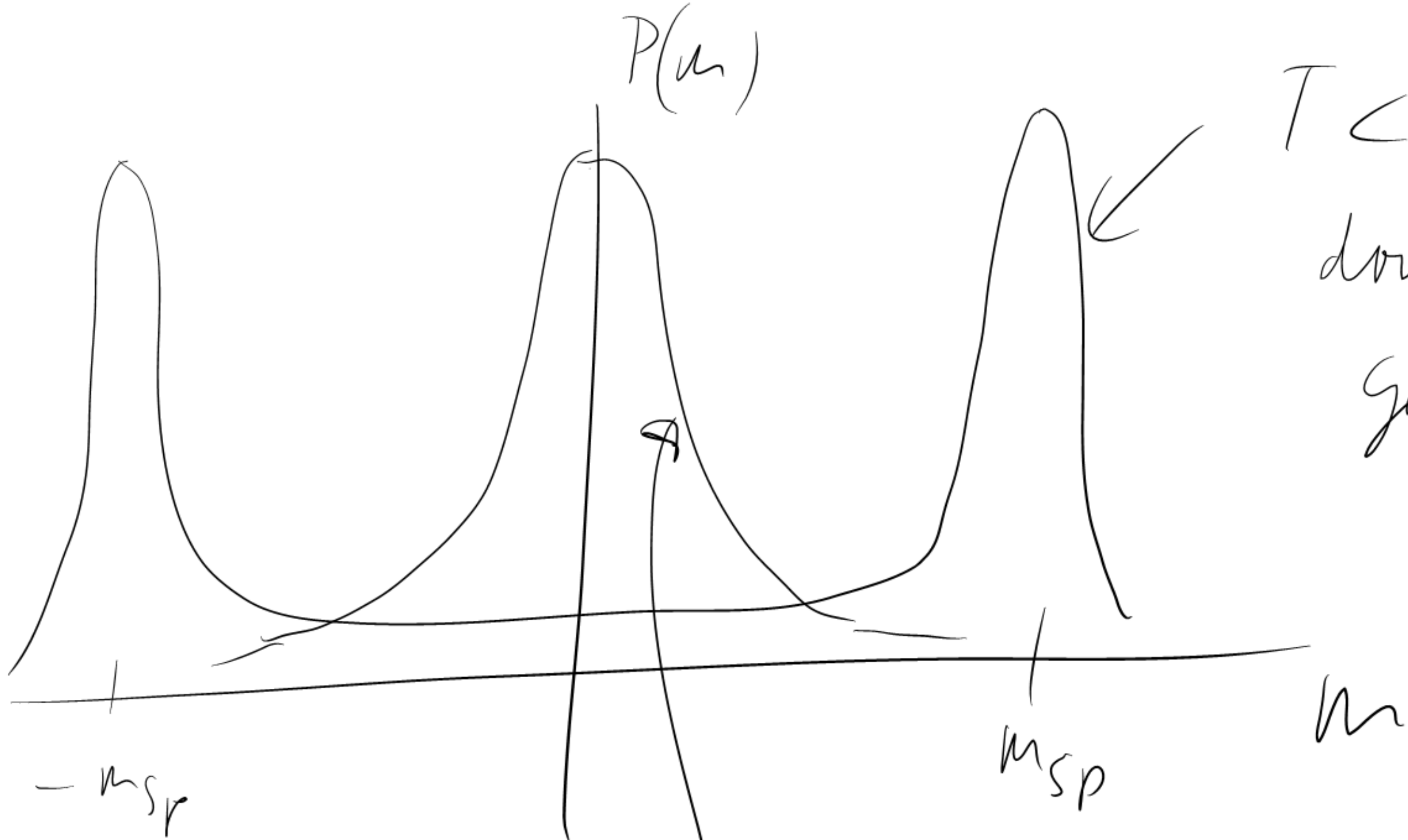


$\xi \propto |t|^{-\nu}$

$\frac{\xi}{L}$ dictates the degree of deviation from asymptotic behavior

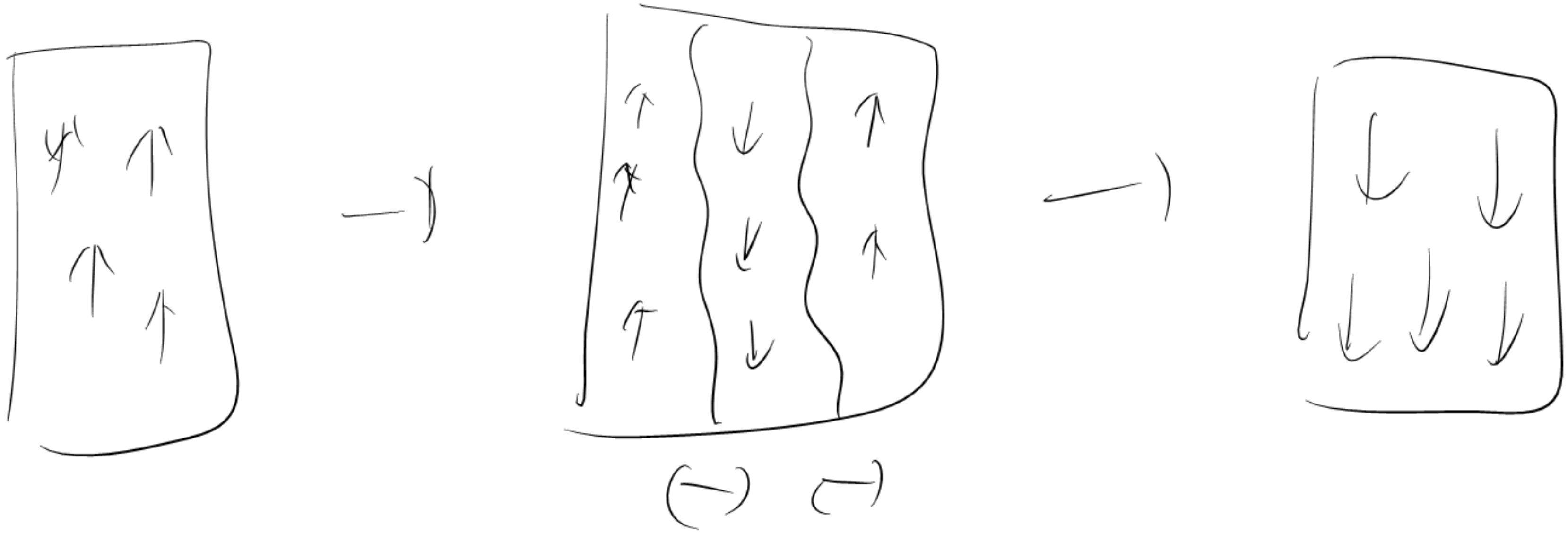
$\rightsquigarrow$ $\boxed{\text{Finite Size Scaling}}$

order parameter distribution



$T << T_c$ double-Gaussian

$T >> T_c$, $P(m)$ Gaussian, $(\text{width})^2 \propto \chi$



FSS: Start: $m \propto |t|^{\beta}$, $\xi \propto |t|^{-\nu}$

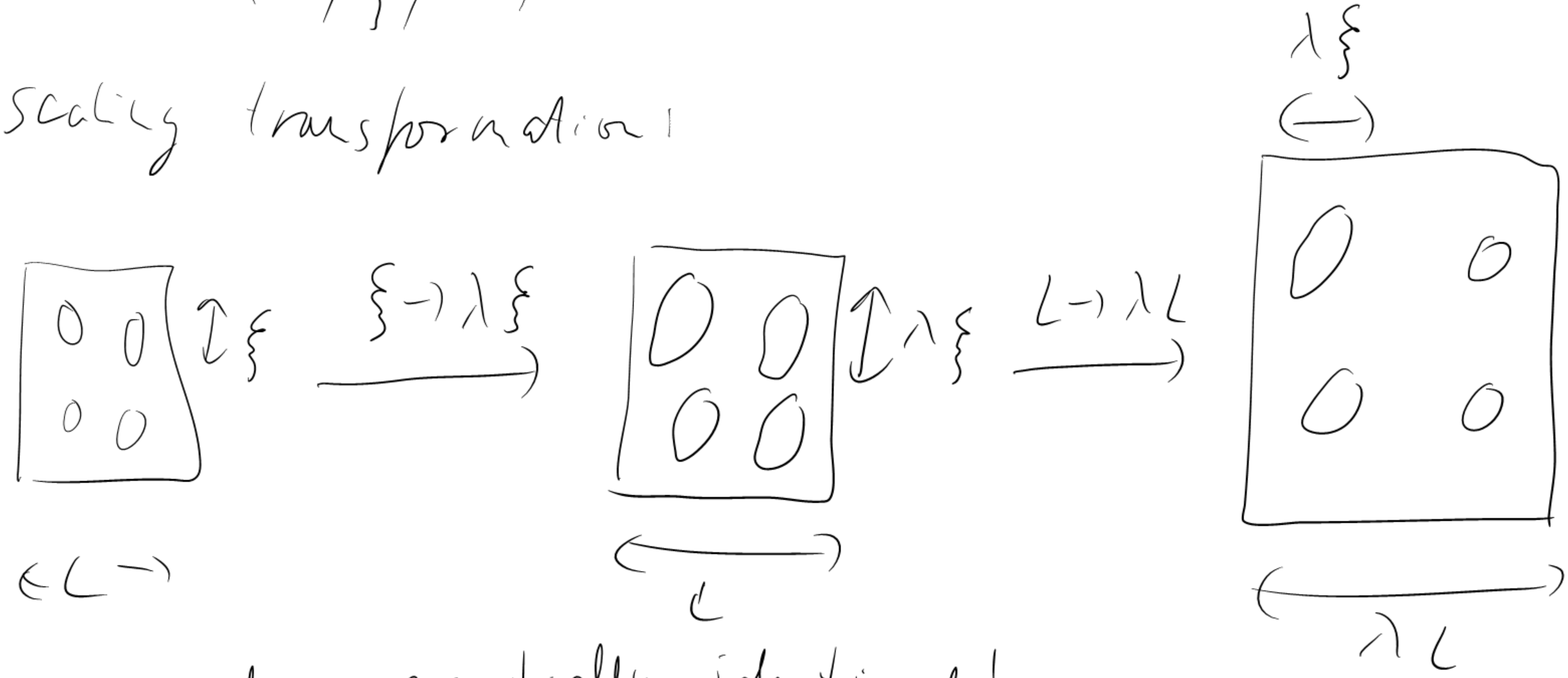
$$m \propto \xi^{-\beta/\nu}$$

"nonlinear thermometer"

$P(m)\,dm \equiv$ probability for finding m in $[m, m+dm]$

$P = P(L, \xi, m)$

scaling transformation:



systems essentially identical!

$$P(\lambda L, \lambda \xi, \lambda^{-\beta/\nu} m)\, d(\lambda^{-\beta/\nu} m) =$$

$$P(L, \xi, m)\, dm \qquad \lambda = L^{-1}$$

$$P(L, \xi, m) = L^{+\beta/\nu}\, \tilde{P}\left(\frac{\xi}{L}, L^{\beta/\nu} m \right)$$

$$\frac{\xi}{L} \sim \frac{t^{-\nu}}{L} = \left[L^{1/\nu} t \right]^{-\nu}$$

$$\boxed{P(L, \xi, m) = L^{\beta/\nu}\, \tilde{\tilde{P}}\left(L^{1/\nu} t, L^{\beta/\nu} m \right)}$$

moments:

$$\langle m^k \rangle = \int_{-\infty}^{\infty} dm \; m^k \; P(L, \xi, m) =$$

$$= \int_{-\infty}^{\infty} dm \; m^k \, L^{\beta/\nu} \, \tilde{P}\left(L^{1/\nu} t, \underbrace{L^{\beta/\nu} m}_{x}\right)$$

$$= L^{-\beta k/\nu} \int_{-\infty}^{\infty} dx \; x^k \; \tilde{P}\left(L^{1/\nu} t, x\right)$$

$$= L^{-\beta k/\nu} \mu_k\left(L^{1/\nu} t\right) = \langle m^k \rangle$$

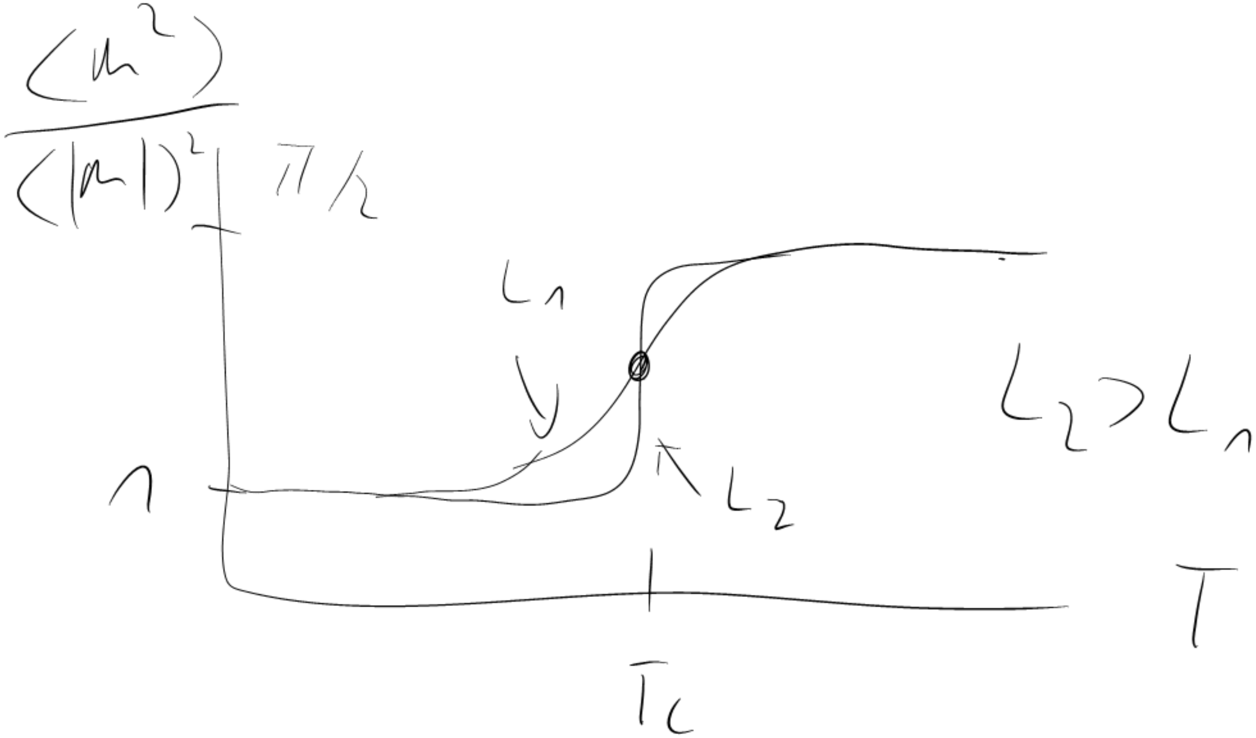
e.g. $$\frac{\langle m^2 \rangle}{\langle |m| \rangle^2} = \frac{L^{-2\beta/\nu}\, \mu_2(L^{1/\nu} t)}{(L^{-\beta/\nu})^2\, \mu_1(L^{1/\nu} t)}$$

$$= \underline{\varphi(L^{1/\nu} t)}$$

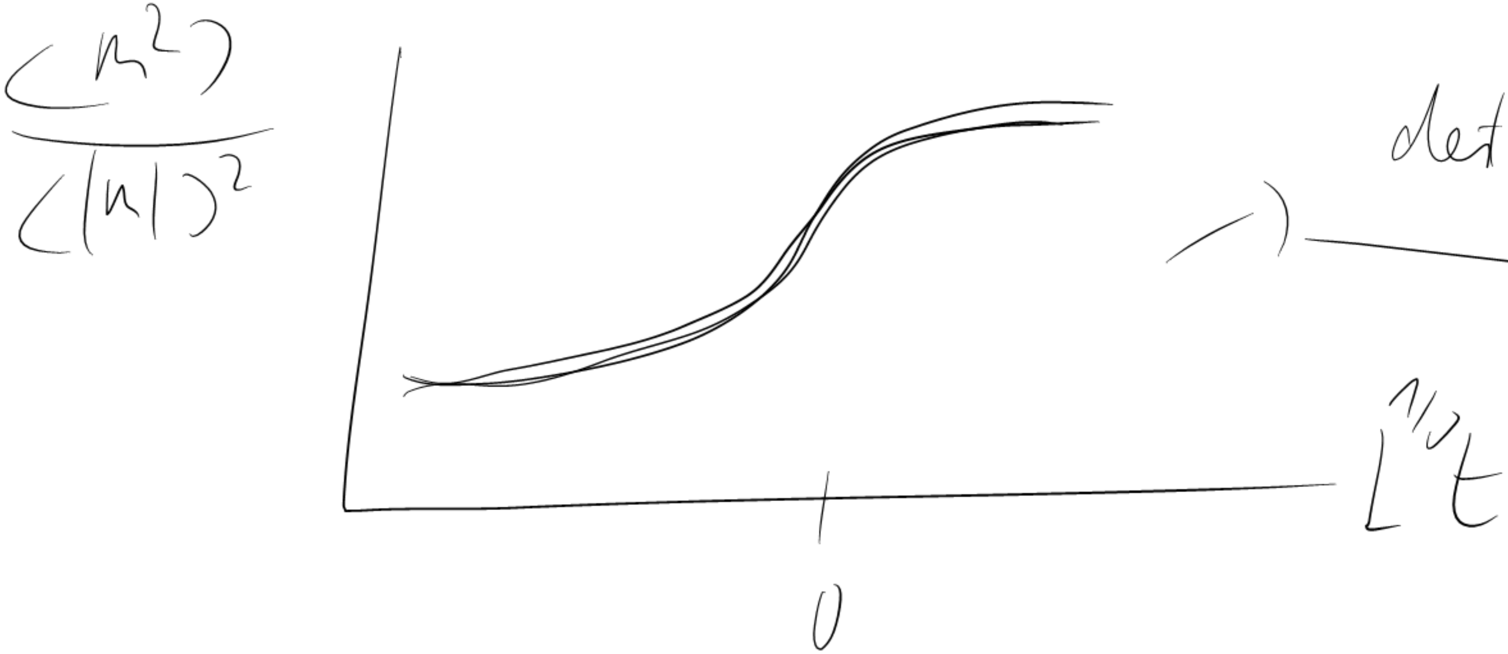
$t = 0 \quad (T = T_c)$

$$\Rightarrow \frac{\langle m^2 \rangle}{\langle |m| \rangle^2} = \varphi(0)$$

regardless of L



$L_2 > L_1 \longrightarrow$ determine T_c



$\longrightarrow$ determine ν

$\langle |m| \rangle (T = T_c) \propto L^{-\beta/\nu} \longrightarrow$ determine β

A. M. Ferrenberg, D. P. Landau, PRB <u>44</u>, 5081 (1991)

$$\frac{J}{T_c} = 0.2216595 \pm 0.0000026$$

$$\nu = 0.6289 \pm 0.0008$$

$$\beta = 0.3258 \pm 0.0044$$

$$\gamma = 1.2390 \pm 0.0071$$

$$\alpha = 2 - d\nu = 0.1133$$

$$\left[\delta = 1 + \frac{\gamma}{\beta} = 4.803\right]$$

SC lattice

3d, nn int.

$T = T_c$

$m \propto H^{1/\delta}$

$\delta_{MF} = 3$

SAW: $R \sim N^{\nu}$

$\nu_{SAW} \sim 0.588$

3D

Li, Madras, Sokal

J. Stat. Phys. (?)

90's