Critical Phenomena

Ising model : $\mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j \qquad S_i = \pm 1$

$J > 0$ nearest-neighbors

Mean Field ~~Theory~~ Approximation

$m \equiv \langle S \rangle \qquad S_i = m + \delta S_i$

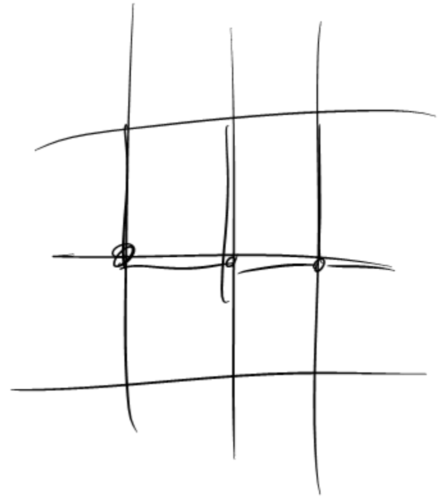
$$S_i S_j = (m + \delta S_i)(m + \delta S_j) = m^2 + m(\delta S_i + \delta S_j) + \underbrace{\delta S_i \delta S_j}_{\text{fluctuation}}$$

const. discard

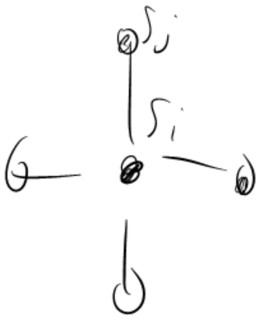
THROW AWAY!

$$\mathcal{H} = -Jm \sum_{(ij)} \left(\delta S_i + \delta S_j\right)$$ add a constant.

$$\mathcal{H} = -Jm \sum_{\langle ij\rangle} \left(S_i + S_j\right)$$



lattice with coordination # z



$z = 4$ # bonds $\equiv N \cdot \frac{z}{2}$

sites

paramagnet

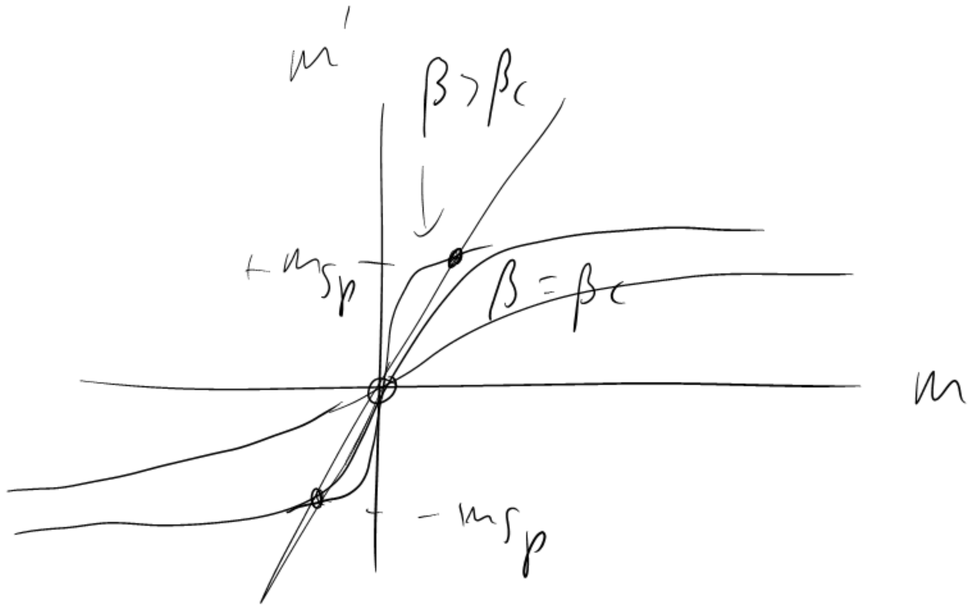
S_i occurs z times
S_j " z "

$$\boxed{\mathcal{H} = -\underbrace{Jmz}_{H} \sum_i S_i}$$

back to paramagnet: $\quad \beta = 1/k_B T$

$$\underbrace{\langle S \rangle}_{= m} = \tanh(\beta H) = \tanh(\beta J z m)$$

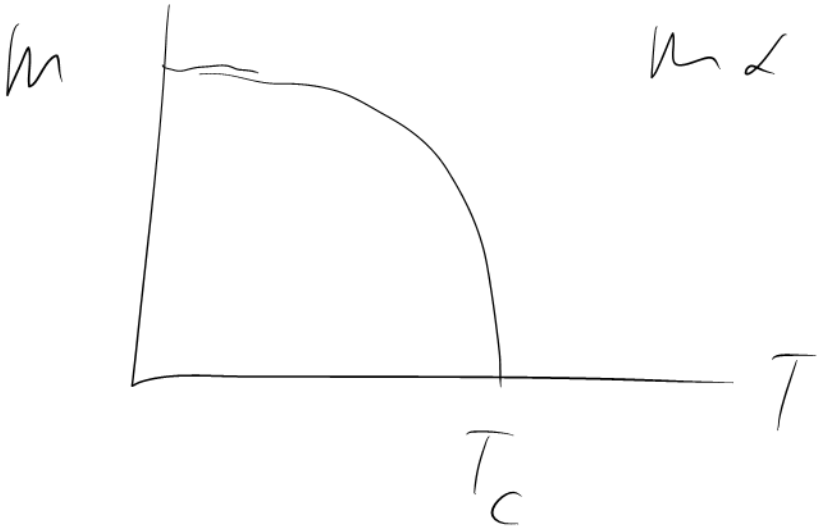
$$\boxed{m = \tanh(\beta J z m)}$$ self-consistency equation



$\beta > \beta_c \Rightarrow$ phase coex.

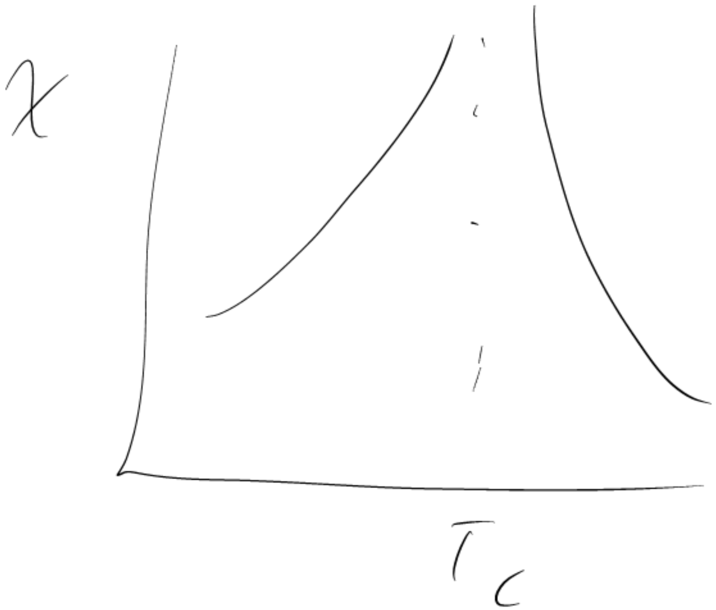
$$1 = \beta_c J z$$

$$\boxed{k_B T_c = J z}$$

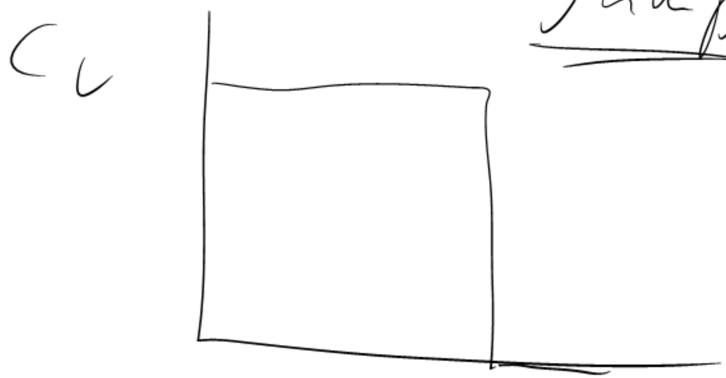


$m \propto |T - T_c|^{1/2}$

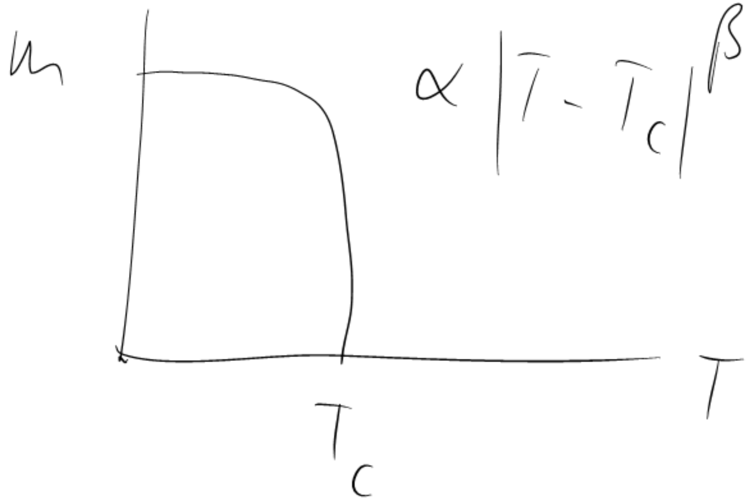
$\chi \propto |T - T_c|^{-1}$



jump.



General Phenomenology



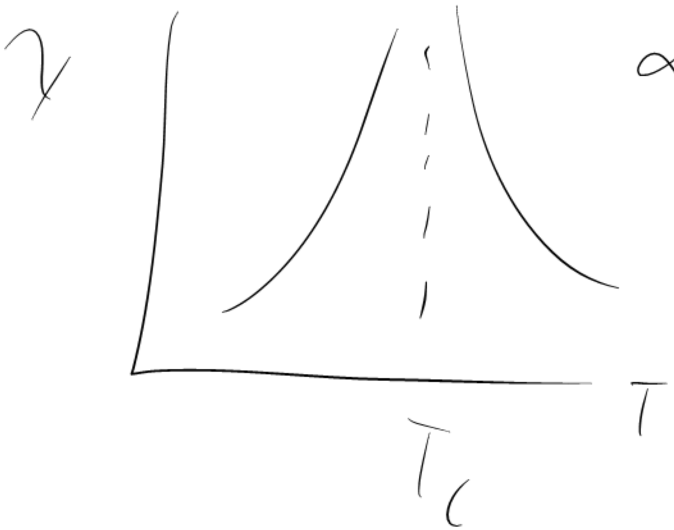
$\propto |T - T_c|^\beta$

"order parameter"

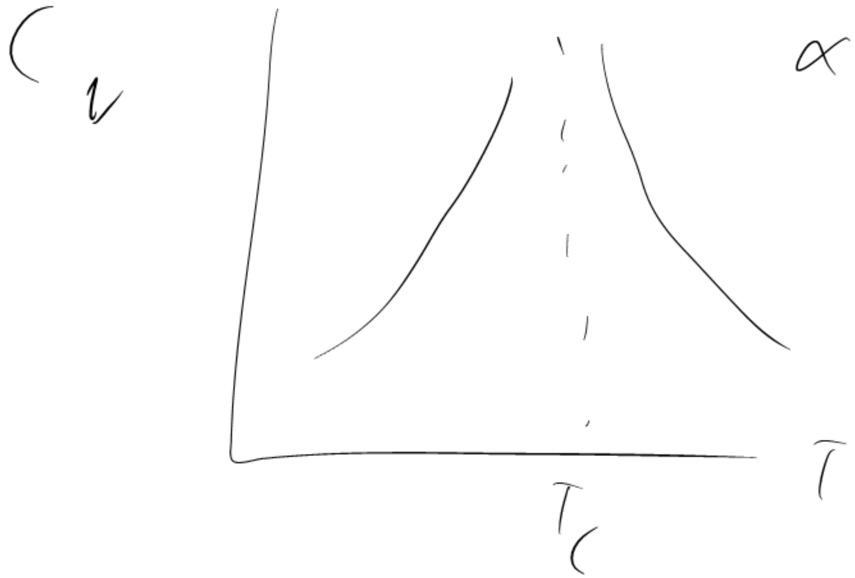
$\equiv 0$ in the high-T phase

$\neq 0$ in the low-T phase

$\beta_{MF} = 1/2$



$\propto |T - T_c|^{-\gamma}$ $\quad \gamma_{MF} = 1$



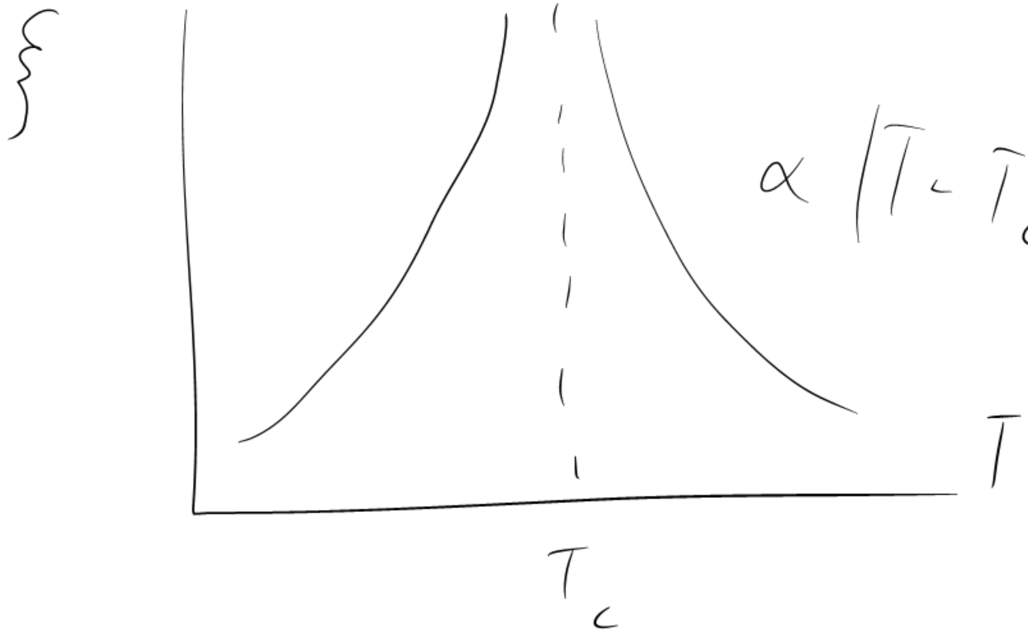
$\propto |T-T_c|^{-\alpha}$

$\alpha_{MF} = 0$

$\langle S(0)\, S(r) \rangle = G(r)$

length scale ξ

= correlation length



$\propto |T-T_c|^{-\nu}$

$\nu_{MF} = 1/2$

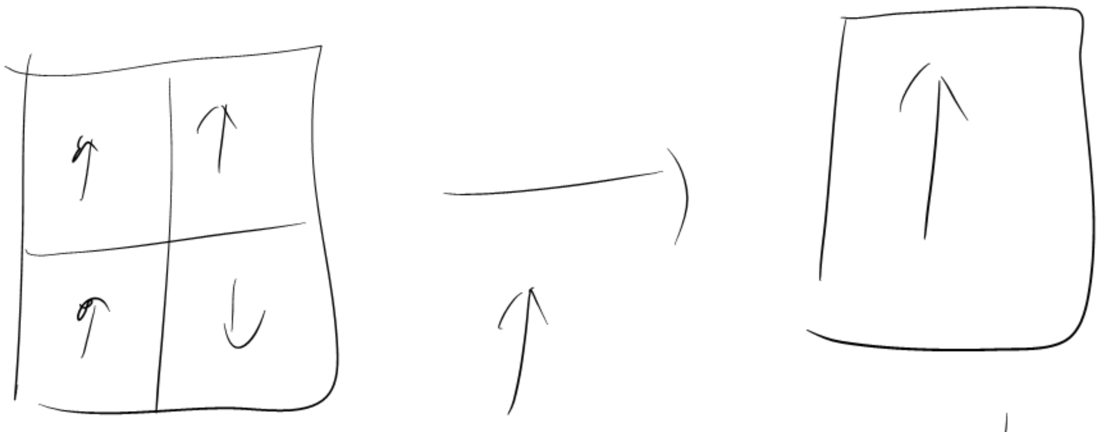
A few concepts

ξ is the central quantity $\equiv$ size of a correlated cluster

$T = T_c$: $\xi = \infty$

at T_c : fluctuations on all length scales
self-similar

renormalization / rescaling

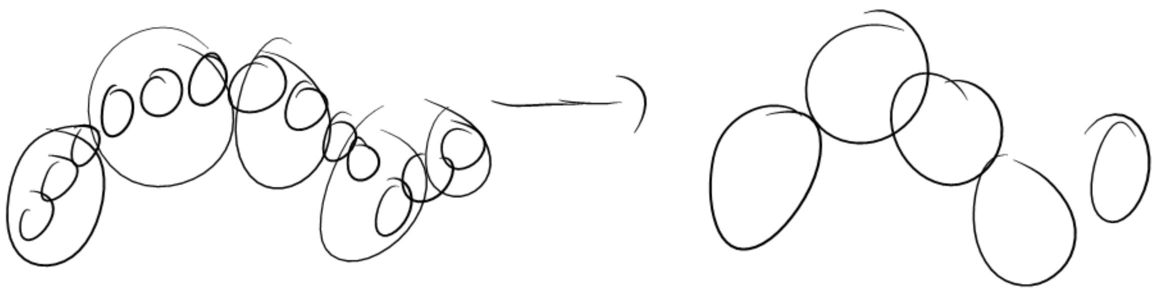


$\mathcal{H}$ coarse-graining $\mathcal{H}'$

"block spin" (Kadanoff)

FIXED POINT HAMILTONIAN

polymers:



many Hamiltonians run into the same $\underbrace{\text{Fixed Point Ham.}}_{FPH}$ → Universality Class!

critical exp. → property of the FPH!

↳ UNIVERSAL

criteria for universality class:

- Spatial dimension
- range of interaction
- symmetry of the Hamiltonian

n-vector model

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

$|\vec{S}_i| = 1$

$\vec{S}_i$ lives in n dimensions

"spin dimension"

$n = 1$ Ising

$n = 2$ XY

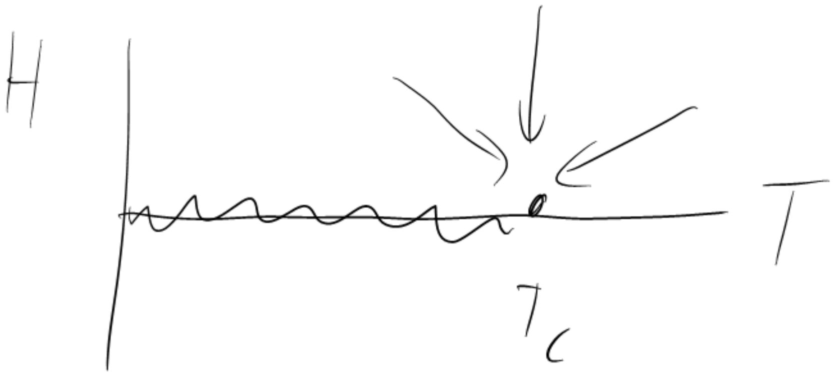
$n = 3$ Heisenberg

$\vdots$

$n \to 0 \to$ polymers (self-avoiding)

(See deGennes, Scaling concepts in Polym. Phys.)

only $\underline{\text{TWO}}$ independent exponents



scaling relations $\rightarrow$ all others

$$G(r) = \langle S(0)\, S(r) \rangle - \langle S \rangle^2$$

near T_c:

$$G(r) = \frac{1}{r^{d-2+\eta}} \tilde{G}\left(\frac{r}{\xi}\right)$$

Meanfield: $\eta = 0$

T_c: $\xi = \infty$

$\tilde{G} = \tilde{G}(0)$

d spatial dimension

lack of energy scale / temperature scale $\Rightarrow$

$\rightsquigarrow$ $\xi \propto |T - T_c|^{-\nu}$

$\rightsquigarrow$ consider ν and η as the fundamental exponents

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Susceptibility: $\chi = \frac{L^d}{k_B T} \left\{ \langle m^2 \rangle - \langle m \rangle^2 \right\}$

$\underbrace{L \times L \times L \times \ldots L}_{d} = L^d$ lattice

$T > T_c \Rightarrow \langle m \rangle = 0$

$$m = \frac{1}{L^d} \underbrace{\sum_i S_i}_{M} = \frac{M}{L^d}$$

$$\chi = \frac{1}{L^d k_B T} \left\{ \langle M^2 \rangle - \langle M \rangle^2 \right\} \overset{T > T_c}{=}$$

$$= \frac{1}{L^d k_B T} \left\langle \left(\sum_i S_i \right)^2 \right\rangle = \frac{1}{L^d k_B T} \sum_{ij} \langle S_i S_j \rangle$$

$$\sim \frac{1}{L^d k_B T} \int d^3 \vec{r} \int d^3 \vec{r}\,' G(\vec{r} - \vec{r}\,') = \frac{1}{k_B T} \int d^3 \vec{r}\, G(\vec{r})$$

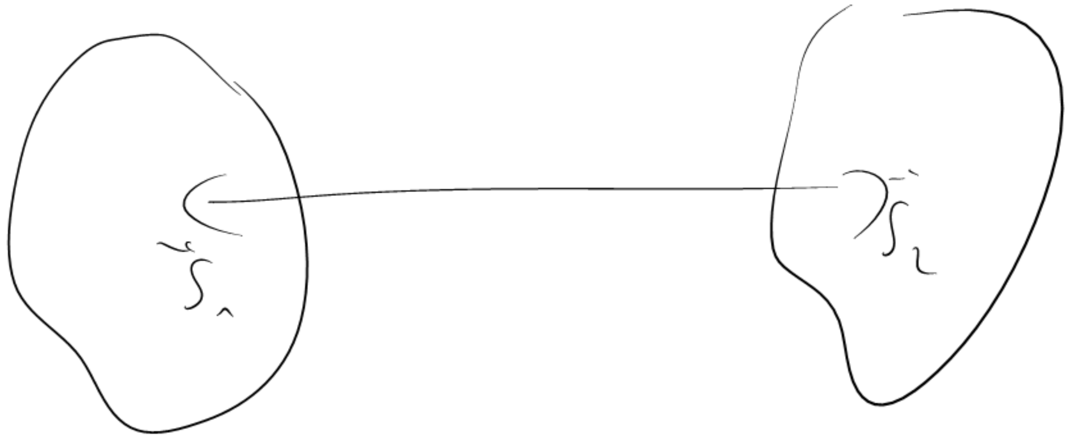
$$\chi \sim \int_0^\infty dr \, r^{d-1} \frac{1}{r^{d-2+\eta}} \tilde{h}\left(\frac{r}{\xi}\right) \qquad x = r/\xi$$

$$\sim \xi^d \frac{1}{\xi^{d-2+\eta}} \underbrace{\int_0^\infty dx \, x^{d-1} \frac{1}{x^{d-2+\eta}} \hat{h}(x)}_{\text{number}}$$

$$\propto \xi^{2-\eta} \propto |t|^{-\nu(2-\eta)} \qquad \boxed{\gamma = \nu(2-\eta)}$$

$$t = \frac{T-T_c}{T_c}$$

$\beta = ?$



$$G \sim (\vec{S}_1 \cdot \vec{S}_2) \sim (\vec{m}_1 \cdot \vec{m}_2)$$

$$\left.\begin{array}{l} \text{decrease } T \Rightarrow \xi \to \lambda \xi \\ \text{increase } r \Rightarrow r \to \lambda r \end{array}\right\} r/\xi \text{ invariant}$$

$$G \longrightarrow \frac{1}{\lambda^{d-2+\eta}} G$$

$$m \longrightarrow \frac{1}{\lambda^{(d-2+\eta)/2}} m$$

on the other hand:

$$t \to \lambda^{-1/\nu} t$$

$$m \to \lambda^{-\beta/\nu} m$$

$$\Rightarrow \frac{\beta}{\nu} = \frac{1}{2}(d-2+\eta)$$

$$\boxed{\beta = \frac{\nu}{2}(d-2+\eta)}$$

$\alpha = ?$

hyperscaling

(cluster)

stores a free energy $O(k_B T)$

total free energy $\simeq \dfrac{L^d}{\xi^d} k_B T$

$$F \propto \xi^{-d} \propto |t|^{+d\nu}$$

$$C_V \propto \frac{\partial^2 F}{\partial T^2} \propto |t|^{-(2-d\nu)}$$

$$\boxed{\alpha = 2 - d\nu}$$ hyperscaling relation

$$\alpha + 2\beta + \gamma = 2 - d\nu + d\nu - 2\nu + \eta\nu + 2\nu - \eta\nu$$

$$= 2$$