$$\delta S = \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial \vec{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{q}}} \right\} \delta \vec{q}_{\parallel}$$

$\Rightarrow \{\ldots\} \perp$ constr. surf. $\Rightarrow \{\ldots\} = \sum_i \lambda_i \frac{\partial G_i}{\partial \vec{q}}$

$$L \to L' = L - \sum_i \lambda_i G_i$$

$$\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j}, \quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j} \right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right)$$

$$\frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial q_j} - \sum_i \lambda_i \frac{\partial G_i}{\partial q_j}$$

$$\Rightarrow \frac{\partial L'}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} - \sum_i \lambda_i \frac{\partial h_i}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

$$\frac{\partial L'}{\partial \dot{\lambda}_i} = 0 \; , \; \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\lambda}_i}\right) = 0 \; , \; \frac{\partial L'}{\partial \lambda_i} = -G_i \; . \; G_i = 0$$

$$\Rightarrow \frac{\partial L'}{\partial \lambda_i} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\lambda}_i}\right) = 0$$

Hamilton Formalism

$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$ canonically conjugate momentum (generalized momentum)

We had: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$

$$\frac{d}{dt} p_i = \dot{p}_i = \frac{\partial L}{\partial q_i} \qquad p_i = p_i(\{q_i\}, \{\dot{q}_i\})$$

2nd order in time, f equations (f: # of degrees of freedom)

$\longrightarrow$ map this onto a system of $2f$ eq. of motion, all 1st order in time

Like: $\ddot{x} = f(x) \iff \begin{cases} \dot{x} = v \\ \dot{v} = f(x) \end{cases}$

BUT: Replace $\dot{q}_i$ by p_i HOW?

Remark: Dimension $(q_i p_i)$ = Js = Dimension (action)

regardless of the nature of the q_i!

Space of the $\{q_i\}, \{p_i\} \equiv$ PHASE SPACE

Legendre transformation!

L : "natural variables" : q_i , $\dot{q}_i$

$$dL = \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \sum_i \frac{\partial L}{\partial q_i} dq_i = \sum_i p_i \underline{d\dot{q}_i} + \sum_i \frac{\partial L}{\partial q_i} \underline{dq_i}$$

We <u>want</u> a function H whose natural variables

$q_i, p_i \Rightarrow \quad dH = \sum_i \overset{?}{\ldots} dp_i + \sum_i \overset{?}{\ldots} dq_i$

Observation: $\quad d(p_i \dot{q}_i) = \dot{q}_i dp_i + p_i d\dot{q}_i$

$$p_i d\dot{q}_i = -\dot{q}_i dp_i + d(p_i \dot{q}_i)$$

$$dL = \sum_i \frac{\partial L}{\partial q_i}\, dq_i - \sum_i \dot{q}_i\, dp_i + \sum_i d(p_i \dot{q}_i)$$

or $$d\left(\sum_i p_i \dot{q}_i - L\right) = \sum_i \dot{q}_i\, dp_i - \sum_i \frac{\partial L}{\partial q_i}\, dq_i$$

<u>Define:</u> $H = \sum_i p_i \dot{q}_i - L$ Hamilton function

$$H = H(\{q_i\}, \{p_i\})$$

$$dH = \sum_i \dot{q}_i\, dp_i - \sum_i \frac{\partial L}{\partial q_i}\, dq_i \Rightarrow$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$-\frac{\partial L}{\partial q_i} = \frac{\partial H}{\partial q_i}$$

eq of motion $\Rightarrow \frac{\partial L}{\partial q_i} = \dot{p}_i \Rightarrow \frac{\partial H}{\partial q_i} = -\dot{p}_i$

$$\Rightarrow \boxed{\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

Hamilton's equation of motion

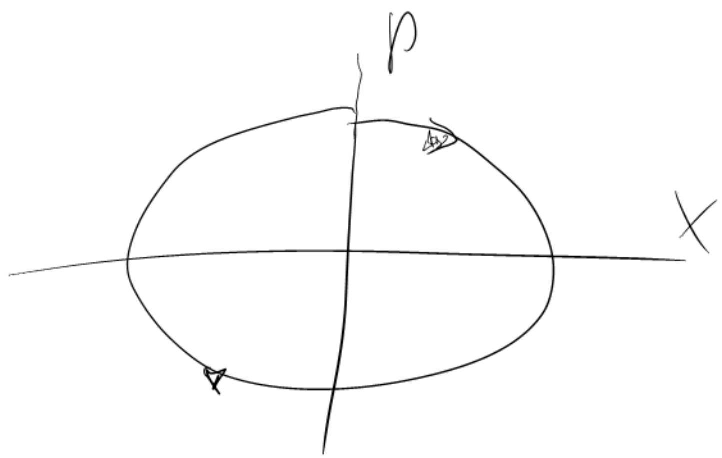
<u>example:</u> harmonic oscillator

$$L = \frac{m}{2}\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 \qquad p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\dot{x} = \frac{1}{m} p \Rightarrow L = \frac{m}{2}\frac{1}{m^2}p^2 - \frac{1}{2}m\omega^2 x^2 = \frac{p^2}{2m} - \frac{m\omega^2}{2}x^2$$

$$H = p\dot{x} - L = p\frac{1}{m}p - \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

$$= \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 = E$$ energy conserved



ellipsis

"energy shell"

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

$$\ddot{x} = \frac{1}{m}\dot{p} = \frac{1}{m}(-m\omega^2 x) = -\omega^2 x$$

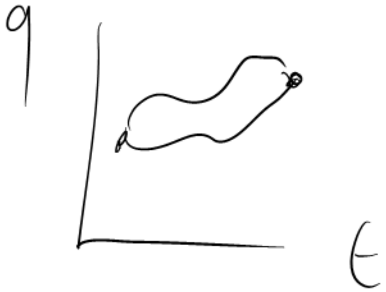
Hamilton's eq. of motion via variational principle

$$S = \int_{t_1}^{t_2} dt\, L = \int_{t_1}^{t_2} dt \left[\sum_i p_i \dot{q}_i - H \right] = S[\vec{q}(t), \vec{p}(t)]$$

↳ $\dot{q}_i(\{p_i\}, \{q_i\})$

↳ depends on the trajectory in phase space

$\delta q_i = 0$ for $t = t_1$, $t = t_2$



NO constraints / conditions on p_i

$$\delta S = \int_{t_1}^{t_2} dt \left\{ \sum_i p_i \delta \dot{q}_i + \sum_i \dot{q}_i \delta p_i - \sum_i \frac{\partial H}{\partial q_i} \delta q_i - \sum_i \frac{\partial H}{\partial p_i} \delta p_i \right\}$$

$$\overset{\text{partial}}{\underset{\text{integr.}}{=}} \int_{t_1}^{t_2} dt \left\{ - \sum_i \dot{p}_i \delta q_i + \sum_i \dot{q}_i \delta p_i - \sum_i \frac{\partial H}{\partial q_i} \delta q_i - \sum_i \frac{\partial H}{\partial p_i} \delta p_i \right\}$$

δq_i, δp_i independent

$$\Rightarrow \quad \dot{p}_i + \frac{\partial H}{\partial q_i} = 0 \,, \quad \dot{q}_i - \frac{\partial H}{\partial p_i} = 0 \quad ✓$$

Poisson Brackets

Observable $f \equiv$ dynamic quantity = function of the phase space variables $f = f(\{q_i\},\{p_i\}, t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial f}{\partial p_i}\dot{p}_i =$$

$$= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} \right)$$

<u>Define</u>: $\{A, B\} := \sum_i \left(\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} \right)$ Poisson bracket

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\}$$ general eq. of motion

<u>properties:</u> $\{A, B\} = -\{B, A\} \Rightarrow \{A, A\} = 0$

elementary PB: $\{q_i, q_j\} = 0 = \{p_i, p_j\}$

$$\{p_i, q_j\} = \delta_{ij}$$

product rule: $\{AB, C\} = A\{B, C\} + \{A, C\}B$

Canonical Transformations $\{p_i\}, \{q_i\} \to \{P_i\}, \{Q_i\}$

(transformation in phase space) is called <u>canonical</u> if

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i} \qquad \dot{Q}_i = \frac{\partial H}{\partial P_i} \qquad H = H(\{P_i\}, \{Q_i\})$$

for <u>any</u> Hamiltonian

Check it out!

$$\dot{Q}_i = \underbrace{\frac{\partial Q_i}{\partial t}}_{=0} + \{H, Q_i\} = \{H, Q_i\}$$

$$\dot{P}_i = \{H, P_i\}$$

$$\{H, Q_i\} = \sum_j \left(\frac{\partial H}{\partial p_j} \frac{\partial Q_i}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial Q_i}{\partial p_j} \right)$$

$$\frac{\partial H}{\partial p_j} = \frac{\partial}{\partial p_j} H(\{Q_k\}, \{P_k\}) = \sum_k \left(\frac{\partial H}{\partial P_k} \frac{\partial P_k}{\partial p_j} + \frac{\partial H}{\partial Q_k} \frac{\partial Q_k}{\partial p_j} \right)$$

$$\frac{\partial H}{\partial q_j} = \sum_k \left(\frac{\partial H}{\partial P_k} \frac{\partial P_k}{\partial q_j} + \frac{\partial H}{\partial Q_k} \frac{\partial Q_k}{\partial q_j} \right)$$

$$\{H, Q_i\} = \sum_{jk} \frac{\partial H}{\partial P_k} \left(\frac{\partial P_k}{\partial p_j} \frac{\partial Q_i}{\partial q_j} - \frac{\partial P_k}{\partial q_j} \frac{\partial Q_i}{\partial p_j} \right)$$

$$+ \sum_{jk} \frac{\partial H}{\partial Q_k} \left(\frac{\partial Q_k}{\partial p_j} \frac{\partial Q_i}{\partial q_j} - \frac{\partial Q_k}{\partial q_j} \frac{\partial Q_i}{\partial p_j} \right) =$$

$$= \sum_k \frac{\partial H}{\partial P_k} \{P_k, Q_i\} + \sum_k \frac{\partial H}{\partial Q_k} \{Q_k, Q_i\}$$

$$\overset{!}{=} \dot{Q}_i = \frac{\partial H}{\partial P_i} \Rightarrow \{P_k, Q_i\} = \delta_{ki} \,, \{Q_k, Q_i\} = 0$$

$\{H, P_i\}$: analogously

$\Rightarrow$ fundamental Poisson brackets must also hold for $\{P_i\}, \{Q_i\}$

(necessary & sufficient for "canonicity")

THEOREM $\{q_i\}, \{p_i\} \rightarrow \{Q_i\}, \{P_i\}$ canonical tr.

$$\{A, B\}_{p,q} = \sum_i \left(\frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} \right)$$

$$\{A, B\}_{P,Q} = \sum_i \left(\frac{\partial A}{\partial P_i} \frac{\partial B}{\partial Q_i} - \frac{\partial A}{\partial Q_i} \frac{\partial B}{\partial P_i} \right)$$

Thm $\{A, B\}_{p,q} = \{A, B\}_{P,Q}$

Proof: Interpret A as a Hamiltonian!

$$\frac{d}{dA} B = \frac{\partial B}{\partial t} + \{A, B\}_{p,q}$$ in p,q-repres

$$\frac{d}{dA} B = \frac{\partial B}{\partial t} + \{A, B\}_{P,Q}$$ in P,Q-repres.

these must be identical!!