Euler Equations & Liouville Operator

Particles:

$$\frac{d}{dt} q_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \frac{d}{dt} p_i = - \frac{\partial \mathcal{H}}{\partial q_i}$$

$$\frac{\partial}{\partial t} P = \mathcal{L} P \qquad \mathcal{L} = \sum_i \left\{ - \frac{\partial}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} + \frac{\partial}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right\}$$

$$= \sum_i \left(- \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial}{\partial q_i} + \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial}{\partial p_i} \right)$$

Fields: $\frac{\partial}{\partial t}\rho = -\partial_\alpha j_\alpha$ $\qquad j_\alpha = \rho u_\alpha$

$$\frac{\partial}{\partial t} j_\alpha = -\partial_\beta \Pi_{\alpha\beta} \qquad \Pi_{\alpha\beta} = p\,\delta_{\alpha\beta} + \underbrace{\rho u_\alpha u_\beta}_{\hat{=}\ \frac{1}{\rho} j_\alpha j_\beta}$$

$$p = p(\rho)$$

$$P = P[\rho, \vec{j}\,]$$

FUNCTIONAL

$$\frac{\partial}{\partial t} P = \mathcal{L} P \qquad \mathcal{L} = ???$$

Functional Derivatives

number

ϕ functional: $\phi[f(x)]$

↳ function

$$\delta\phi = \int dx \, \frac{\delta\phi}{\delta f(x)} \, \delta f(x) \qquad f(x) \to f(x) + \delta f(x)$$

↳ functional derivative

in linear approximation

especially: $\delta f(x) = \varepsilon \, \delta(x - x_0)$

$$\to \delta\phi = \varepsilon \frac{\delta\phi}{\delta f(x_0)} \qquad \delta\phi = \phi[f(x) + \varepsilon\,\delta(x - x_0)] - \phi[f(x)]$$

$$\frac{\delta \phi}{\delta f(x_0)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \phi\left[f(x) + \varepsilon \delta(x - x_0) \right] - \phi\left[f(x) \right] \right\}$$

$$f(x) = \int dx' \, \delta(x - x') \, f(x')$$

$$\delta f(x) = \int dx' \, \delta(x - x') \, \delta f(x')$$

$$\boxed{\frac{\delta f(x)}{\delta f(x')} = \delta(x - x')}$$

$$\frac{\partial}{\partial t}\rho = -\partial_\alpha j_\alpha \qquad \frac{\partial}{\partial t} j_\alpha = -\partial_\beta \Pi_{\alpha\beta} \quad \Rightarrow$$

$$\mathcal{L} = -\int d^3\vec{r}\, \frac{\delta}{\delta\rho}\left(-\partial_\alpha j_\alpha\right) + \frac{\delta}{\delta j_\alpha}\left(-\partial_\beta \Pi_{\alpha\beta}\right)$$

(from Kramers-Moyal)

$$= \int d^3\vec{r}\,(\partial_\alpha j_\alpha)\frac{\delta}{\delta\rho} + \int d^3\vec{r}\,\frac{\delta}{\delta j_\alpha}\left(\partial_\beta \Pi_{\alpha\beta}\right)$$

$$= \int d^3\vec{r}\,(\partial_\alpha j_\alpha)\frac{\delta}{\delta\rho} + \int d^3\vec{r}\int d^3\vec{r}\,'\,\delta(\vec{r}-\vec{r}\,')\frac{\delta}{\delta j_\alpha(\vec{r})}\left(\partial_\beta' \Pi_{\alpha\beta}(\vec{r}\,')\right) =$$

$$= \int d^3\vec{r}\,(\partial_\alpha j_\alpha)\frac{\delta}{\delta\rho} + \int d^3\vec{r}\int d^3\vec{r}\,'\,\delta(\vec{r}-\vec{r}\,')\left[\partial'_\beta\,\Pi_{\alpha\beta}(\vec{r}\,')\right]\frac{\delta}{\delta j_\alpha(\vec{r}\,')}$$

$$+ \underbrace{\int d^3\vec{r}\int d^3\vec{r}\,'\,\delta(\vec{r}-\vec{r}\,')\,\partial'_\beta\left\{\frac{\partial\Pi_{\alpha\beta}}{\partial j_\alpha}(\vec{r}\,')\,\delta(\vec{r}-\vec{r}\,')\right\}}$$

= last term =

$$= -\int d^3\vec{r}\int d^3\vec{r}\,'\left\{\frac{\partial\Pi_{\alpha\beta}}{\partial j_\alpha}(\underbrace{\vec{r}\,'}_{\vec{r}})\,\delta(\vec{r}-\vec{r}\,')\right\}\underbrace{\partial'_\beta}_{-\partial_\beta}\,\delta(\vec{r}-\vec{r}\,') =$$

$$= +\int d^3\vec{r}\int d^3\vec{r}\,'\left\{\frac{\partial\Pi_{\alpha\beta}}{\partial j_\alpha}(\vec{r})\,\delta(\vec{r}-\vec{r}\,')\right\}\partial_\beta\,\delta(\vec{r}-\vec{r}\,') =$$

$$= - \int d^3\vec{r} \int d^3\vec{r}\,' \, \delta(\vec{r} - \vec{r}\,') \, \partial_\beta \left\{ \frac{\partial \Pi_{\alpha\beta}}{\partial j_\alpha}(\vec{r}) \, \delta(\vec{r} - \vec{r}\,') \right\} \overset{\vec{r} \leftrightarrow \vec{r}\,'}{=}$$

$$= - \int d^3\vec{r} \int d^3\vec{r}\,' \, \delta(\vec{r} - \vec{r}\,') \, \partial'_\beta \left\{ \frac{\partial \Pi_{\alpha\beta}}{\partial j_\alpha}(\vec{r}\,') \, \delta(\vec{r} - \vec{r}\,') \right\} =$$

$= -$ last term $\Rightarrow$ last term $= 0$!!

$$\mathcal{L} = \int d^3\vec{r} \, (\partial_\alpha j_\alpha) \frac{\delta}{\delta \rho} + \int d^3\vec{r} \, (\partial_\beta \Pi_{\alpha\beta}) \frac{\delta}{\delta j_\alpha}$$

$$\mathcal{H} = \int d^3\vec{r} \left\{ \underbrace{\frac{1}{2} \frac{j_\alpha j_\alpha}{\rho}}_{\text{kinetic energy}} + \underbrace{e(\rho)}_{\text{(potential) energy density}} \right\}$$

$$L\mathcal{H} = ??$$

$$\frac{\delta \mathcal{H}}{\delta \rho} = \frac{1}{2} j_\alpha j_\alpha \left(-\frac{1}{\rho^2}\right) + \frac{\partial e}{\partial \rho} = -\frac{1}{2} u_\alpha u_\alpha + \frac{\partial e}{\partial \rho}$$

$$\frac{\delta \mathcal{H}}{\delta j_\alpha} = \frac{1}{2\rho} 2 j_\alpha = u_\alpha$$

$$L\mathcal{H} = \int d^3\vec{r}\,(\partial_\beta \Pi_{\alpha\beta})\,u_\alpha$$

$$-\frac{1}{2}\int d^3\vec{r}\,u_\alpha u_\alpha\,\partial_\beta(\rho u_\beta)$$

$$+\int d^3\vec{r}\,\frac{\partial e}{\partial \rho}\,\partial_\alpha(\rho u_\alpha)$$

Relation $e \leftrightarrow p$?

1st law: $dE = T\underbrace{dS}_{=0} - p\,dV + \mu \underbrace{dN}_{=0}$

M total mass

$$p = -\frac{\partial E}{\partial V}\Big|_S \qquad E = eV \qquad \frac{\partial E}{\partial V} = e + V\frac{\partial e}{\partial V} =$$

$$= e + V\frac{\partial e}{\partial\left(\frac{M}{\rho}\right)} = e + \frac{V}{M}\frac{\partial e}{\partial \rho}\frac{\partial \rho}{\partial(1/\rho)} =$$

$$= e + \frac{1}{\rho}\frac{\partial e}{\partial \rho}\left(\frac{\partial(1/\rho)}{\partial \rho}\right)^{-1} = e + \frac{1}{\rho}\frac{\partial e}{\partial \rho}(-\rho^2) = e - \rho\frac{\partial e}{\partial \rho} = -p$$

$$\boxed{p = \rho \frac{\partial e}{\partial \rho} - e}$$

3rd term $= \int d^3\vec{r}\, \frac{\partial e}{\partial \rho} \partial_\alpha (\rho u_\alpha) =$

$$= \int d^3\vec{r}\, \frac{\partial e}{\partial \rho} \rho\, \partial_\alpha u_\alpha + \int d^3\vec{r}\, u_\alpha \frac{\partial e}{\partial \rho} \partial_\alpha \rho =$$

$$= \int d^3\vec{r}\, (p+e)\, \partial_\alpha u_\alpha + \int d^3\vec{r}\, u_\alpha\, \partial_\alpha e =$$

$$= \int d^3\vec{r}\, (p+e)\, \partial_\alpha u_\alpha - \int d^3\vec{r}\, e\, \partial_\alpha u_\alpha =$$

$$= \int d^3\vec{r}\, p\, \partial_\alpha u_\alpha = -\int d^3\vec{r}\, u_\alpha\, \partial_\alpha p =$$

$$= -\int d^3\vec{r}\, u_\alpha\, \partial_\beta \left(p\,\delta_{\alpha\beta} + \rho u_\alpha u_\beta - \rho u_\alpha u_\beta \right) =$$

$$= -\int d^3\vec{r}\, u_\alpha\, \partial_\beta \Pi_{\alpha\beta} + \int d^3\vec{r}\, u_\alpha\, \partial_\beta (\rho u_\alpha u_\beta) =$$

$$= -\int d^3\vec{r}\, u_\alpha\, \partial_\beta \Pi_{\alpha\beta} - \int d^3\vec{r}\, \rho u_\alpha u_\beta\, \partial_\beta u_\alpha$$

$$\text{2nd term} = -\frac{1}{2} \int d^3\vec{r}\, u_\alpha u_\alpha\, \partial_\beta (\rho u_\beta) =$$

$$= +\frac{1}{2} \int d^3\vec{r}\, \rho u_\beta\, \underbrace{\partial_\beta (u_\alpha u_\alpha)}_{2 u_\alpha \partial_\beta u_\alpha} = \int d^3\vec{r}\, \rho u_\alpha u_\beta\, \partial_\beta u_\alpha$$

$$\Rightarrow \mathcal{L}\mathcal{H} = \int d^3\vec{r}\; u_\alpha \,\partial_\beta \Pi_{\alpha\beta}$$

$$+ \int d^3\vec{r}\; \rho u_\alpha u_\beta \,\partial_\beta u_\alpha$$

$$- \int d^3\vec{r}\; u_\alpha \,\partial_\beta \Pi_{\alpha\beta}$$

$$- \int d^3\vec{r}\; \rho u_\alpha u_\beta \,\partial_\beta u_\alpha$$

$= 0 \quad \Rightarrow$ energy is CONSERVED!

$$\mathcal{L}\, e^{-\beta\mathcal{H}} = -\beta e^{-\beta\mathcal{H}} \mathcal{L}\mathcal{H} = 0$$

Fluctuating Hydrodynamics

So far: $\mathcal{L} = \mathcal{L}_H$ ← Hamiltonian part of Fokker-Planck operator

add dissipation: $\mathcal{L} = \mathcal{L}_H + \mathcal{L}_D$

$$\mathcal{L}_D = -\int d^3\vec{r} \, \frac{\delta}{\delta j_\alpha} \left(\partial_\beta \, \eta_{\alpha\beta\gamma\delta} \, \partial_\gamma u_\delta \right) =$$

$$= -\eta_{\alpha\beta\gamma\delta} \int d^3\vec{r} \, \frac{\delta}{\delta j_\alpha} \left(\partial_\beta \partial_\gamma u_\delta \right)$$

add noise: $\mathcal{L} = \mathcal{L}_H + \mathcal{L}_D + \mathcal{L}_N$

$$\mathcal{L}_N = + \int d^3\vec{r} \int d^3\vec{r}\,' \frac{\delta}{\delta j_\alpha(\vec{r})} \frac{\delta}{\delta j_\gamma(\vec{r}\,')} k_B T \, \eta_{\alpha\beta\gamma\delta}$$

$$\partial_\beta \underbrace{\partial'_\delta}_{-\partial_\delta} \delta(\vec{r} - \vec{r}\,') =$$

$$= -k_B T \, \eta_{\alpha\beta\gamma\delta} \int d^3\vec{r} \int d^3\vec{r}\,' \frac{\delta}{\delta j_\alpha(\vec{r})} \frac{\delta}{\delta j_\gamma(\vec{r}\,')} \partial_\beta \partial_\delta \delta(\vec{r} - \vec{r}\,')$$

$$= -k_B T \, \eta_{\alpha\beta\gamma\delta} \int d^3\vec{r} \frac{\delta}{\delta j_\alpha(\vec{r})} \partial_\beta \partial_\delta \frac{\delta}{\delta j_\gamma(\vec{r})}$$

symmetric

$$\mathcal{L}_N = -k_B T\, \eta_{\alpha\beta\gamma\delta} \int d^3\vec{r}\, \frac{\delta}{\delta j_\alpha(\vec{r})}\, \partial_\beta \partial_\gamma \frac{\delta}{\delta j_\delta(\vec{r})}$$

$$\frac{\delta}{\delta j_\delta(\vec{r})} e^{-\beta \mathcal{H}} = -\beta e^{-\beta\mathcal{H}} \frac{\delta \mathcal{H}}{\delta j_\delta(\vec{r})} =$$

$$= -\beta e^{-\beta\mathcal{H}} u_\delta$$

$$\mathcal{L}_N e^{-\beta\mathcal{H}} = \eta_{\alpha\beta\gamma\delta} \int d^3\vec{r}\, \frac{\delta}{\delta j_\alpha(\vec{r})}\, \partial_\beta \partial_\gamma u_\delta\, e^{-\beta\mathcal{H}}$$

$$\mathcal{L}_D e^{-\beta\mathcal{H}} = -\eta_{\alpha\beta\gamma\delta} \int d^3\vec{r}\, \frac{\delta}{\delta j_\alpha(\vec{r})}\, \partial_\beta \partial_\gamma u_\delta\, e^{-\beta\mathcal{H}}$$

$$(\mathcal{L}_N + \mathcal{L}_D)\, e^{-\beta \mathcal{H}} = 0 \;, \quad \mathcal{L}_H\, e^{-\beta \mathcal{H}} = 0$$

$$\boxed{\mathcal{L}\, e^{-\beta \mathcal{H}} = 0}$$

$\boxed{\text{FDT HOLDS!!}}$