Markov process $\longrightarrow$ generalized FPE

$$\frac{\partial}{\partial t} P = \mathcal{L} P \;, \quad \mathcal{L} = \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n D^{(n)}$$

$$D^{(n)} = \underline{D^{(n)}(x,t)} = \lim_{\tau \to 0} \frac{1}{n!\,\tau} \langle (\Delta x)^n \rangle$$

FPE:

$$\mathcal{L} = -\frac{\partial}{\partial x} D^{(1)} + \frac{\partial^2}{\partial x^2} D^{(2)}$$

<u>many</u> variables

$$\mathcal{L} = -\sum_i \frac{\partial}{\partial x_i} D^{(1)}_i + \sum_{i} \frac{\partial^2}{\partial x_i \partial x_j} D^{(2)}_{ij}$$

drift coefficient $\to$ vector

$$D_i^{(1)} = \frac{\langle \Delta x_i \rangle}{\tau} \qquad \tau \to 0$$

diffusion coefficient $\to$ 2nd rank tensor

$$D_{ij}^{(2)} = \frac{1}{2\tau} \langle \Delta x_i \, \Delta x_j \rangle \qquad \tau \to 0$$

$$P(x, t \,|\, x_0, t_0) = \exp\left(\mathcal{L}(t - t_0)\right) \, \delta(x - x_0)$$

for homogeneous processes ($D^{(n)}$ do not depend explicitly on time)

$$\boxed{x_i(t+\tau) = x_i(t) + D_i^{(1)}\tau + \sqrt{2\tau}\, r_i}$$

$$\Delta x_i = x_i(t+\tau) - x_i(t) = D_i^{(1)}\tau + \sqrt{2\tau}\, r_i$$

$$\langle \Delta x_i \rangle = D_i^{(1)}\tau$$ ✓

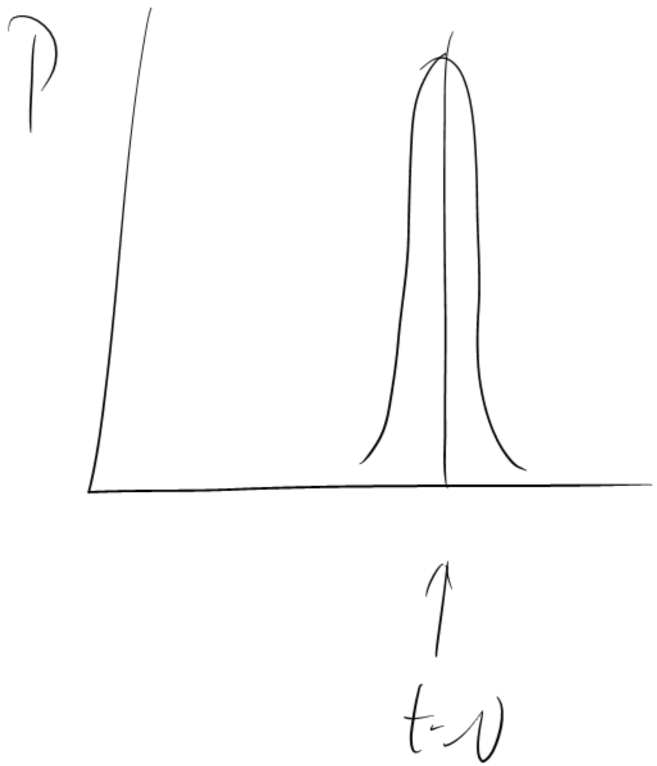
$$\langle \Delta x_i \Delta x_j \rangle = \underset{o(\tau)}{D_i^{(1)} D_j^{(1)} \tau^2} + \underset{D_{ij}}{\langle r_i r_j \rangle} 2\tau$$

$$+ D_i^{(1)} \underbrace{\langle r_j \rangle}_{=0} \tau \sqrt{2\tau} + D_j^{(1)} \underbrace{\langle r_i \rangle}_{=0} \tau \sqrt{2\tau}$$

$$= 2\tau D_{ij} + o(\tau)$$ ✓

$$\langle \Delta x_i \, \Delta x_j \, \Delta x_k \rangle = O(\tau^{3/2}) + \text{higher order}$$

$$\langle (\Delta x_i)^n \rangle = O(t^{n/2}) + \text{"}$$



$$\mathcal{L} = \frac{\partial^2}{\partial x^2} D^{(2)} - \frac{\partial}{\partial x} D^{(1)}$$

$$D^{(2)} \downarrow \text{const.} \qquad D^{(1)} \downarrow \text{const.}$$

$P(x,t) \equiv P(x,t\,|\,0,0)$

$P(x,0) = \delta(x)$

$$P(x,t) = \int_{-\infty}^{+\infty} dk\, \tilde{P}(k,t) \exp(ikx)$$

$$\frac{\partial}{\partial t}\tilde{P} = \left[-k^2 D^{(2)} - ikD^{(1)}\right]\tilde{P} \Rightarrow \tilde{P}(k,t) = \tilde{P}(k,0) \times$$

$$\times \exp\left\{\left(-k^2 D^{(2)} - ikD^{(1)}\right)t\right\}$$

$$\tilde{P}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, P(x,t) \exp[-ikx]$$

$$\tilde{P}(k,0) = \frac{1}{2\pi} \Rightarrow \tilde{P}(k,t) = \frac{1}{2\pi} \exp\left\{\left[-ikD^{(1)} - k^2 D^{(2)}\right] t\right\}$$

$$P(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, \exp\left\{ ik(x - D^{(1)}t) - k^2 D^{(2)} t \right\} \overset{\text{Gaussian}}{=} \dots$$

$$\dots = \frac{1}{\sqrt{4\pi D^{(2)} t}} \exp\left\{ - \frac{(x - D^{(1)} t)^2}{4 D^{(2)} t} \right\}$$

$$\langle x \rangle = D^{(1)} t \qquad \langle (x - \langle x \rangle)^2 \rangle = 2 D^{(2)} t$$

$$\underbrace{x_i(t+\tau) = x_i(t) + D_i^{(1)}\tau + \sqrt{2\tau}\, r_i} \qquad | : \tau$$

$$\frac{1}{\tau}\left[x_i(t+\tau) - x_i(t)\right] = D_i^{(1)} + \underbrace{\sqrt{\frac{2}{\tau}}}_{\to\infty} r_i \qquad \underline{\underline{\tau \to 0}}$$

$$\frac{d}{dt} x_i = D_i^{(1)} + f_i(t)$$

$\hookrightarrow$ Gaussian white noise

"stochastic differential equation"

$$\langle f_i \rangle = 0$$

$$\langle f_i(t) f_j(t') \rangle = 2 D_{ij}^{(2)} \delta(t - t')$$

$$\Delta x_i = D_i^{(1)} \tau + \underbrace{\int_0^\tau dt \, f_i(t)}_{\sim \sqrt{2\tau}\, r_i} = D_i^{(1)} \tau + \Delta x_i^{st}$$

$$\langle \Delta x_i^{st} \Delta x_j^{st} \rangle = \int_0^t dt_1 \int_0^\tau dt_2 \langle f_i(t_1) f_j(t_2) \rangle = \int_0^t dt_1 \int_0^\tau dt_2 \, 2 D_{ij}^{(2)} \delta(t_1 - t_2) = 2 D_{ij}^{(2)} \tau$$

back to diffusion & drift

$$\frac{\partial}{\partial t} P = \left[D^{(2)} \frac{\partial^2}{\partial x^2} - D^{(1)} \frac{\partial}{\partial x} \right] P$$

Langevin: $$\frac{d}{dt} x = D^{(1)} + f(t)$$

$$\langle f(t) f(t') \rangle = 2 D^{(2)} \delta(t - t')$$ integrate:

$$x(t) = x(0) + D^{(1)} t + \underbrace{\int_0^t d\tau\, f(\tau)}_{\text{Gaussian random variable}}$$

$\rightarrow$ $x(t)$ is GAUSSIAN

$$P(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\xi)^2}{2\sigma^2}\right) \qquad \xi = ? \quad \sigma^2 = ?$$

$$\xi(t) = \langle x(t) \rangle \qquad \sigma^2(t) = \langle (x - \langle x \rangle)^2 \rangle$$

$$\langle x(t) \rangle = x(0) + D^{(1)} t \qquad x(0) = 0$$

$$x - \langle x \rangle = \int_0^t d\tau \, f(\tau)$$

$$\langle (x - \langle x \rangle)^2 \rangle = \int_0^t d\tau_1 \int_0^t d\tau_2 \, \underbrace{\langle f(\tau_1) f(\tau_2) \rangle}_{2 D^{(2)} \delta(\tau_1 - \tau_2)}$$

$$= 2 D^{(2)} t = \sigma^2$$

$$P(x,t) = \frac{1}{\sqrt{4\pi D^{(2)} t}} \exp\left\{ - \frac{(x - D^{(1)} t)^2}{4 D^{(2)} t} \right\}$$

back to **Momentum relaxation**

Gaussian white noise

$$\frac{d}{dt} p = -\frac{\xi}{m} p + f(t)$$

unknown strength of noise

$$\langle f(t) \rangle = 0, \quad \langle f(t) f(t') \rangle = 2 Q \, \delta(t-t')$$

homogeneous eq.:

$$\frac{d}{dt} p = -\frac{\xi}{m} p \Rightarrow p(t) = p(0) \exp\left(-\frac{\xi}{m} t\right)$$

inhomog.: "variation of constants"

$$p(t) = \pi(t) \exp\left(-\frac{\xi}{m} t\right) \Rightarrow \quad \Pi(0) = p(0) = p_0$$

$$\frac{d}{dt} p = \frac{d\pi}{dt} \exp\left(-\frac{\xi}{m} t\right) + \pi(t) \exp\left(-\frac{\xi}{m} t\right)\left(-\frac{\xi}{m}\right)$$

$$= -\frac{\xi}{m} p + \exp\left(-\frac{\xi}{m} t\right) \frac{d\pi}{dt} = -\frac{\xi}{m} p + f(t)$$

$$\frac{d}{dt} \Pi = \exp\left(+\frac{\xi}{m} t\right) f(t)$$

$$\pi(t) = \Pi(0) + \int_0^t d\tau \exp\left(+\frac{\xi}{m} \tau\right) f(t)$$

$$p(t) = p_0 \exp\left(-\frac{\xi}{m} t\right) + \int_0^t d\tau \exp\left(-\frac{\xi}{m}(t-\tau)\right) f(\tau)$$

→ $p(t)$ is a linear combination of Gaussian random variables

→ $p(t)$ is GAUSSIAN! ONLY unknown par. are $\langle p(t) \rangle$, $\langle [p(t) - \langle p(t) \rangle]^2 \rangle$

$$\langle p(t) \rangle = p_0 \exp\left(-\frac{\xi}{m} t\right)$$

$$\left\langle \left(p - \langle p\rangle\right)^2 \right\rangle = \int_0^t d\tau_1 \int_0^t d\tau_2 \, \exp\left(-\frac{\xi}{m}\left(2t - \tau_1 - \tau_2\right)\right)$$

$$\underbrace{\left\langle f(\tau_1) f(\tau_2) \right\rangle}_{2Q\,\delta(\tau_1 - \tau_2)}$$

$$= 2Q \int_0^t d\tau_1 \exp\left[-\frac{\xi}{m}\left(2t - 2\tau_1\right)\right] =$$

$$= 2Q \exp\left(-\frac{2\xi}{m} t\right) \frac{\exp\left(+\frac{2\xi}{m} t\right) - 1}{\frac{2\xi}{m}} =$$

$$= \underbrace{\frac{mQ}{\xi}\left(1 - \exp\left(-\frac{2\xi}{m}t\right)\right)} = \text{variance}$$

$$t \to \infty \Rightarrow \langle (p - \langle p \rangle)^2 \rangle \longrightarrow \frac{mQ}{\xi}$$

Statistical mechanics:

process describes relaxation into thermal equilibrium $\to$ p should be Maxwell-distributed for $t \to \infty$

equip. th. $\frac{\langle p^2 \rangle}{2m} = \frac{T}{2} \Rightarrow \langle p^2 \rangle = \underline{mT}$

$$\Rightarrow \frac{m Q}{\xi} = m T \Rightarrow \boxed{\boxed{Q = T \xi}}$$

"fluctuation-dissipation theorem"

$$\frac{\partial}{\partial t} P = \mathcal{L} P \qquad \frac{d}{dt} p = -\frac{\xi}{m} p + f$$

$$\mathcal{L} = -\frac{\partial}{\partial p}\left(-\frac{\xi}{m}\right) p + \frac{\partial^2}{\partial p^2} Q = \frac{\xi}{m}\frac{\partial}{\partial p}\left(p + \frac{Qm}{\xi}\frac{\partial}{\partial p}\right)$$

$$\langle f(t) f(t') \rangle = 2Q\,\delta(t-t')$$

$$\frac{\partial}{\partial p} \exp(-\beta \mathcal{H}) = \frac{\partial}{\partial p} \exp\left(-\frac{p^2}{2mT}\right) = -\frac{p}{mT} \exp\left(-\frac{p^2}{2mT}\right)$$

$$\mathcal{L} \exp(-\beta \mathcal{H}) = \frac{\xi}{m}\frac{\partial}{\partial p}\left(p - \frac{Qm}{\xi}\frac{p}{mT}\right) \exp\left(-\frac{p^2}{2mT}\right)$$

$$= \frac{\xi}{m}\frac{\partial}{\partial p} p\left(1 - \frac{Q}{\xi T}\right) \exp\left(-\frac{p^2}{2mT}\right) =, \quad \boxed{Q = \xi T}$$