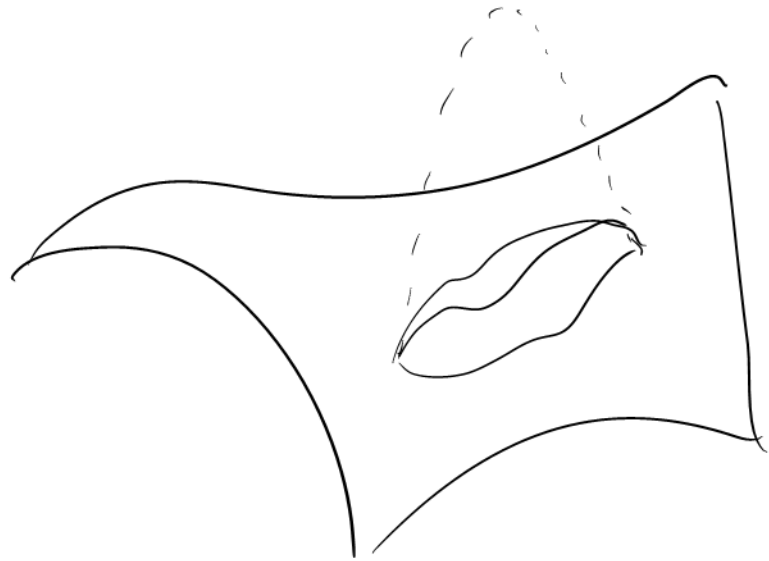


$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \quad L = L(\{q_i\}, \{\dot{q}_i\})$$



const. surf. $G = 0$

Lagrange multipliers: First for simple

calculus, $f(x_1, x_2, \dots, x_N) \stackrel{!}{=} \text{Min. with}$

CONSTRAINT $G(x_1, x_2, \dots, x_N) = 0$

$$d\vec{x} = \underbrace{d\vec{x}_{\parallel}}_{\substack{\text{in the} \\ \text{surface}}} + \underbrace{dx_{\perp}}_{\substack{\text{perpendic.} \\ \text{to the surface}}}$$

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i = (\vec{\nabla} f) \cdot d\vec{x} = (\vec{\nabla} f) \cdot [d\vec{x}_{\parallel} + d\vec{x}_{\perp}]$$

$d\vec{x}$ should ONLY live within the surface!

$$d\vec{x}_{\perp} = 0 \quad \text{that means: } df = 0 \\ \Rightarrow (\vec{\nabla} f) \cdot d\vec{x}_{\parallel} = 0$$

$\vec{\nabla} f \perp \text{surface}$. We know: $\vec{\nabla} G \perp \text{surface!}$

$$\vec{\nabla} f = \lambda \vec{\nabla} G \quad \uparrow \text{Lagrange multiplier}$$

$$\phi(x_1, x_2, \dots, x_N, \lambda) = f(x_1, \dots, x_N) - \lambda G(x_1, \dots, x_N)$$

$$\frac{\partial \phi}{\partial x_i} = \frac{\partial f}{\partial x_i} - \lambda \frac{\partial G}{\partial x_i} \stackrel{!}{=} 0 \quad \Rightarrow \quad \vec{\nabla} f = \lambda \vec{\nabla} G \quad \checkmark$$

$$\frac{\partial \phi}{\partial \lambda} = -G(x_1, \dots, x_N) \stackrel{!}{=} 0 \quad \Rightarrow \quad G = 0$$

Calculus of variations:

$$S = \int_{t_1}^{t_2} dt L(\{q_i\}, \{\dot{q}_i\}) \stackrel{!}{=} \text{Min.}$$

$$\delta S(\{q_i\}) = 0 \quad \delta S = \sum_i \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i$$

$$= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial \vec{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{q}}} \right] \cdot \left[\underbrace{\delta \vec{q}_{||}}_{=} + \underbrace{\delta \vec{q}_{\perp}}_{=} \right]$$

$$\int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial \vec{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{q}}} \right] \cdot \delta \vec{q}_{||} = 0$$

= 0

$$\Rightarrow \frac{\partial L}{\partial \vec{q}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) \perp \text{surface}$$

$$\Rightarrow \frac{\partial L}{\partial \vec{q}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) = \lambda \vec{\nabla} G$$

Several constraints: $\frac{\partial L}{\partial \vec{q}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) = \sum_i \lambda_i \vec{\nabla} G_i$

$$L \rightarrow L' = L - \sum_i \lambda_i G_i$$

$$L' = L'(\{q_i\}, \{\dot{q}_i\}, \{\lambda_i\})$$

$$\frac{\partial L'}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}_i} \right) = \frac{d}{dt} \frac{\partial}{\partial \dot{q}_i} (L - \sum_j \lambda_j h_j)$$

$$= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \sum_j \lambda_j \frac{d}{dt} \frac{\partial}{\partial \dot{q}_i} h_j$$

$$\frac{\partial L'}{\partial \lambda_i} = \frac{d}{dt} \underbrace{\frac{\partial L'}{\partial \dot{\lambda}_i}}_{=0} = 0 = -h_i \Rightarrow \underbrace{h_i = 0}$$

Noether, Theorem

relation symmetries \leftrightarrow conservation laws

continuous symmetry $S(\alpha)$
transformation \leftarrow \uparrow parameter

$$q_i(t) \xrightarrow{S(\alpha)} q_i'(t) = q_i(t, \alpha)$$

$$q_i(t, 0) = q_i(t)$$

$$\alpha = 0 \quad (\Leftrightarrow) \quad S(\alpha) = \mathbb{1}$$

Requirement: L is invariant under S

$$q_i \longrightarrow q_i' = q_i(t, \alpha)$$

$$L \longrightarrow L$$

$$L(\{q_i(t, 0)\}, \{\dot{q}_i(t, 0)\}) = L(\{q_i(t, \alpha)\}, \{\dot{q}_i(t, \alpha)\})$$

$$\frac{d}{d\alpha} L = 0 \quad \text{or} \quad 0 = \sum_i \left[\frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \alpha} \right] \stackrel{\text{eq.}}{=} \text{mot.}$$

$$= \sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial q_i} \frac{d}{dt} \frac{\partial q_i}{\partial \alpha} \right] =$$

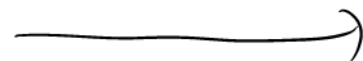
$$= \frac{d}{dt} \sum_i \left[\frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \alpha} \right] = 0$$

$$\Rightarrow \sum_i \left[\frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \alpha} \right] = \text{const.}$$

conservation
law related
to $S(\alpha)$

Translation in Space

$$\vec{r}_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$$



$$\begin{pmatrix} x_i + \alpha \\ y_i \\ z_i \end{pmatrix} = \vec{r}_i'$$

$$\dot{\vec{r}}_i \rightarrow \dot{\vec{r}}_i \quad L \rightarrow L$$

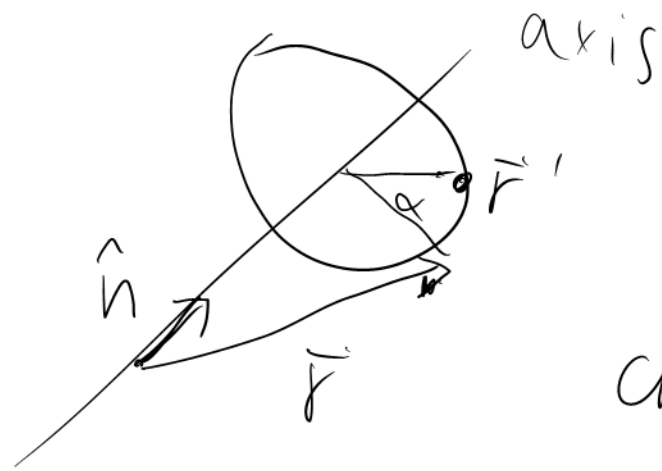
corresponds to conservation of momentum:

$$\text{const.} \equiv P_x = \sum_i \left[\frac{\partial L}{\partial \dot{x}_i} \underbrace{\frac{\partial x_i}{\partial \alpha}}_{=1} + \frac{\partial L}{\partial \dot{y}_i} \underbrace{\frac{\partial y_i}{\partial \alpha}}_{=0} + \frac{\partial L}{\partial \dot{z}_i} \underbrace{\frac{\partial z_i}{\partial \alpha}}_{=0} \right]$$

$$= \sum_i \frac{\partial L}{\partial \dot{x}_i} = \sum_i (\text{particle momentum})$$

$$\Rightarrow \vec{P} = \text{const.}$$

Rotation in space



\hat{n} : unit vector
along the axis

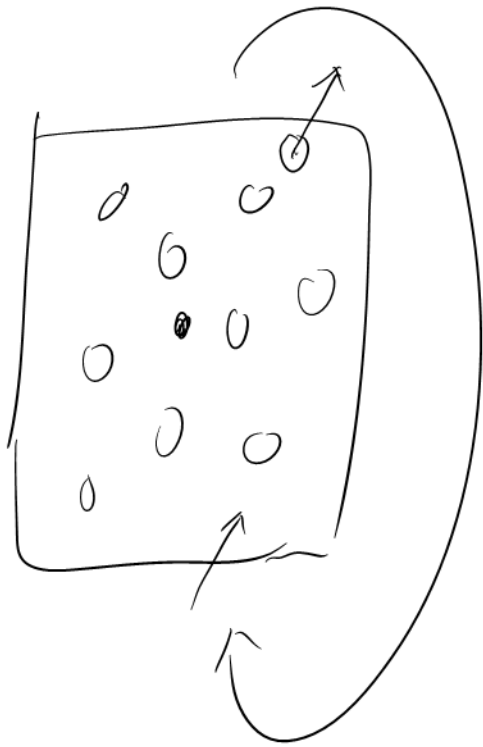
α : angle

$$\text{const.} = \sum_i \underbrace{\frac{\partial L}{\partial \dot{\vec{r}}_i}}_{\vec{p}_i} \underbrace{\frac{\partial \vec{r}_i}{\partial \alpha}}_{= \hat{n} \times \vec{r}_i} =$$

$$= \sum_i \vec{p}_i \cdot (\hat{n} \times \vec{r}_i) = \hat{n} \cdot \underbrace{\sum_i (\vec{r}_i \times \vec{p}_i)}_{=\vec{L} \text{ angular mom.}}$$

$$\text{const.} = \hat{n} \cdot \vec{L} \quad \hat{n} \text{ arbitrary}$$

$$\Rightarrow \vec{L} = \text{const.}$$



Molecules

Periodic
boundary
conditions

Dynamics

Gauge Invariance

$$U \longrightarrow U + \underbrace{\Delta U}_{\text{constant}}$$

$$m_i \ddot{r}_i = - \frac{\partial U}{\partial \vec{r}_i}$$

$$L \longrightarrow L - \Delta U$$

More general: $L \longrightarrow L + \frac{d}{dt} \Psi(\{q_i\}, t)$

→ dynamics is not changed

for any
function Ψ

WHY?

$$S = \int_{t_1}^{t_2} dt \mathcal{L}(\{q_i\}, \{\dot{q}_i\}, t)$$

$$\rightarrow \int_{t_1}^{t_2} dt \left(\mathcal{L} + \frac{d}{dt} \Psi \right) = S + \Psi \Big|_{t_1}^{t_2} =$$

$$= S - \Psi(\{q_i(t_1)\}, t_1) + \Psi(\{q_i(t_2)\}, t_2) =$$

$$= S - \Psi(\{q_i\}, t_1) + \Psi(\{q_i\}, t_2)$$

FIXED

does not depend on trajectory

THEOREM. We search for an M :

$L \rightarrow L + M$: Dynamics is unchanged

for ANY Lagrangian L : M MUST HAVE

the form $M = \frac{d}{dt} \psi(\{q_i\}, t)$

Proof : $\frac{d}{dt} \left(\frac{\partial M}{\partial \dot{q}} \right) = \frac{\partial M}{\partial q}$ For ANY trajectory $q(t)$

(just ONE degree of freedom)

Ansatz: $M = a_0(q, t) + a_1(q, t) \dot{q} + a_2(q, t) \frac{1}{2} \dot{q}^2 + \dots$

$$\frac{\partial M}{\partial \dot{q}} = a_1(q, t) + a_2(q, t) \dot{q} + \dots$$

$$\frac{d}{dt} \left(\frac{\partial M}{\partial \dot{q}} \right) = \frac{\partial a_1}{\partial q} \dot{q} + \frac{\partial a_1}{\partial t} + \frac{\partial a_2}{\partial q} \dot{q}^2 + \frac{\partial a_2}{\partial t} \dot{q} + a_2 \ddot{q} + \dots =$$

$$= \dots + \ddot{q} \left(a_2 + a_3 \dots + a_4 \dots + a_5 \dots \right)$$

Should be ZERO

otherwise: $\ddot{q} = \dots \rightarrow$ one trajectory and not ∞ many!

$$M = a_0(q, t) + a_1(q, t) \dot{q}$$

$$\frac{\partial M}{\partial \dot{q}} = a_1 \quad \frac{d}{dt} \frac{\partial M}{\partial \dot{q}} = \frac{\partial a_1}{\partial q} \dot{q} + \frac{\partial a_1}{\partial t}$$

$$\frac{\partial M}{\partial q} = \frac{\partial a_0}{\partial q} + \frac{\partial a_1}{\partial q} \dot{q}$$

$$\left. \begin{array}{l} \frac{\partial a_1}{\partial t} = \frac{\partial a_0}{\partial q} \end{array} \right\}$$

Dy. $\phi(q, t) = \int dq a_1(q, t) \Rightarrow a_1 = \frac{\partial \phi}{\partial q}$

$$\frac{\partial a_1}{\partial t} = \frac{\partial^2 \phi}{\partial q \partial t} = \frac{\partial^2 \phi}{\partial t \partial q} = \frac{\partial a_0}{\partial q} \Rightarrow a_0 = \frac{\partial \phi}{\partial t} + c(t)$$

$$\underline{\text{Def.}}: \quad \Psi = \phi + \int dt \, c(t)$$

$$\frac{\partial \Psi}{\partial t} = \frac{\partial \phi}{\partial t} + c(t) \Rightarrow a_0 = \frac{\partial \Psi}{\partial t}$$

$$\frac{\partial \Psi}{\partial q} = \frac{\partial \phi}{\partial q} = a_1$$

$$\Rightarrow M = a_0 + a_1 \dot{q} = \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial q} \dot{q} = \frac{d}{dt} \Psi$$



Generalized Noether - Theorem

$$q_i(t) \xrightarrow{S(\alpha)} q_i(t, \alpha)$$

$$S(\alpha = 0) = \mathbb{1}$$

L is NOT invariant, but rather

$$L \xrightarrow{S(\alpha)} L + \alpha \frac{d}{dt} \Psi(\{q_i\}, t) + O(\alpha^2)$$

$$\frac{dL}{d\alpha} = \frac{d}{dt} \Psi \quad \left[\text{compared to } \frac{dL}{d\alpha} = 0 \right]$$

\rightarrow does not change the dynamics

$$\frac{dL}{d\alpha} = \frac{d}{dt} \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \alpha} \right) = \frac{d}{dt} \psi$$

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \alpha} - \psi = \text{const.}$$

Galilei Transform

$$\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$$

→

$$\begin{pmatrix} x_i - vt \\ y_i \\ z_i \end{pmatrix}$$

constant
velocity

$$\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \longrightarrow \dot{x}_i'^2 + \dot{y}_i'^2 + \dot{z}_i'^2$$

$$\dot{x}_i'^2 = \dot{x}_i^2 + v^2 - 2v\dot{x}_i = \dot{x}_i^2 + \frac{d}{dt} [v^2 t - 2v x_i]$$

$$L \longrightarrow L + \frac{d}{dt} \sum_i m_i \frac{1}{2} (v^2 t - 2v x_i)$$

$$v \longrightarrow 0:$$

$$L \longrightarrow L - v \underbrace{\frac{d}{dt} \sum_i m_i x_i}_{-4}$$

$$\frac{\partial L}{\partial \dot{x}_i} = p_{ix} \quad \frac{\partial x_i}{\partial V} = -t$$

$$\text{const.} = \sum_i p_{ix} (-t) + \sum_i m_i x_i \quad \left| : \underbrace{\sum_i m_i}_{=M} \right. \\ \text{total mass}$$

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \quad \text{center of mass}$$

$$\text{const.} = -t \frac{p_x}{M} + R_x$$

$$R_x = \text{const.} + t \frac{p_x}{M}$$