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Solution of the Adjoint Problem for Instabilities with a Deformable Surface: Rosensweig and Marangoni Instability

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We present a method to find the adjoint system of equations and the corresponding boundary conditions for free deformable surfaces. Motivated by the nonlinear discussion of the Rosensweig instability in ferrogels using the energy method, we treat the surface as dynamic and take the stationary limit only in the very end. We analyze the adjoint system of dynamic equations together with its corresponding boundary conditions and present as a solution the adjoint eigenvectors for the Rosensweig instability. The method is also applied to pure surface tension driven convection (Marangoni convection). [DOI: 10.1063/1.2757709]

I. INTRODUCTION

One prominent example of a surface phenomenon is the normal-field or Rosensweig instability in magnetic fluids[1]. Magnetic fluids are colloidal suspensions of ferromagnetic nanoparticles in a carrier liquid[2]. If an external magnetic field, applied perpendicularly to the free surface of a magnetic fluid, exceeds a certain critical value, the initially flat surface becomes unstable in favor of a stationary pattern of hexagonal surface spikes in two dimensions. The same kind of instability occurs, when the properties of a magnetic fluid are combined with those of a polymeric gel to form a ferrogel. However, the critical magnetization of the ferrogel is enhanced by the shear modulus of the elastic network[3].

In the case of usual ferrofluids, the instability has been carefully studied experimentally[4], a satisfying theoretical description in terms of a multiple scale analysis as done for the Rayleigh-Bénard instability[5] is, however, not yet available. Early weakly nonlinear discussions[6,7] used an energy minimizing method, that was valid only in the case of a vanishing magnetic susceptibility. Later on this method was extended to magnetic fluids with a finite but still small magnetic susceptibility[8] and to magnetic gels[9]. Nevertheless, this method, as well as the functional analysis method used by Twombly and Thomas[10] and Silber and Knobloch[11], is only valid in the static regime and gives no information about the growth of the surface perturbations towards a final static pattern and furthermore neglect any dissipative processes that might be of importance during the growth of surface spikes. The dynamics of walls between hexagonal and square patterns has been analyzed first in Refs. 12, 13 using a Swift-Hohenberg equation. However, a multiple scale analysis based on the fundamental hydrodynamic equations is still missing. Malik and Singh[14,15] applied an $\epsilon-$expansion to the hydrodynamic equations, where ϵ denotes the difference between the actual applied magnetic field and the critical one. They circumvented Fredholm's theorem and, even more importantly, they restricted the discussion to potential flows only. Lange[16] first mentioned that the adjoint system of the linear fundamental equations and especially the corresponding boundary conditions required to use Fredholm's theorem is still missing in the case of the Rosensweig instability. However, the use of a scalar product, introduced for the Marangoni instability[17] provided the free surface is undeformable, did not yield the adjoint system.

The Marangoni instability is another prominent example of a surface tension driven instability. If a temperature gradient is applied to a layer of a fluid with a free surface, the conducting state becomes unstable beyond a certain critical temperature gradient when heating is done from below and convection starts. For thick layers the instability is driven by buoyancy (classical Rayleigh-Bénard convection), but if the layer is smaller than about 1mm, Pearson[18] proposed fluctuations of the surface tension, due to temperature fluctuations at the free surface, being the mechanism driving the convection.

This instability was investigated extensively theoretically. Nield[19] first compared linearly the competition between the buoyancy and the surface tension driven instability, but both, Pearson and Nield, still considered a flat, undeformable surface. Scriven and Sternling[20] and later on Smith[21] accounted for a free deformable surface. In Ref. 20 capillary effects have been considered, but an always unstable conducting regime was obtained due to missing gravitational contributions. Smith discussed a layer model, a light fluid above a heavier one. But a comprehensive linear study was first given by Takashima[22,23], who also discussed the possibility of an oscillatory branch that could arise for negative Marangoni numbers. Pérez-García and Carneiro[24] generalized this approach to the combination of both, surface driven and bouyancy driven convection, which matches the results of Takashima in the limit of negligible buoyancy forces. This nonlinear theoretical discussion assumed a flat, undeformable surface as did all the other approaches. For example, Rosenblat *et al.* discussed the nonlinear regime in a cylindrical container[25] in

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terms of an extended Galerkin method where no adjoint system was used. This discussion was later on extended to rectangular vessels[26,27]. The case of a horizontally infinite layer of fluid was studied in Refs. 17, 28. In Ref. 29 a two layer model was considered, where the adjoint system was derived using the ansatz of[17] provided the surface is flat. An adjoint system, based on the fact that the surface is deformable, is therefore also in the case of the Bénard-Marangoni instability highly desirable.

In this work we present a method to find the adjoint operator and its corresponding adjoint boundary conditions taking into account the deformations of a free surface. First we discuss the case of a magnetic gel in an external magnetic field. Later on we will apply that method to the case of Bénard-Marangoni convection in usual fluids.

II. SURFACE WAVES

The general idea of our approach is to treat the surface as dynamic with surface waves propagating on the free surface, as long as the magnetic field – or the temperature gradient in the case of convection – is below its critical value. We can distinguish the limiting cases of capillary waves for very short wavelengths, gravitational waves for rather long wavelengths and Rayleigh elastic waves in the intermediate regime and only in the case of gels. Furthermore, below the critical point of the instability, all of these waves are damped, but get excited again by thermal agitation. When reaching the critical value of the control parameter, the damping of one characteristic mode becomes weak and finally vanishes exactly at the critical point. In the stationary case this coincides with the slowing down of this particular mode, so that the initially traveling waves transform into a static pattern. This process can be seen by inspection of the dispersion relation, e.g. in Ref. 2 in case of pure ferrofluids.

As a consequence of this, we assume the entire linear problem to be time dependent from the beginning. Only in the end of the discussion we will, based on the discussion of the dispersion relation, take the stationary limit of the system.

III. BASIC EQUATIONS AND GROUND STATE

The dynamic equations have already been given in our linear discussion of the Rosensweig instability. They have been derived for the general case by Jarkova et al.[30] and are repeated here using some approximations discussed below

$$\partial_t g_i + \partial_j T_{ij} = \rho G_i \quad (1)$$

$$(\partial_t + v_k\partial_k)\epsilon_{ij} - \frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right) = 0 \quad (2)$$

$$\partial_t \rho + \partial_k(\rho v_k) = 0 \quad (3)$$

The equations describe the conservation of linear momentum $\mathbf{g} = \rho\mathbf{v}$ (1) and of mass ρ (3), whereas eq. (2) accounts for the dynamic strain field which is related to the broken translational symmetry of the system. An index i denotes the ith

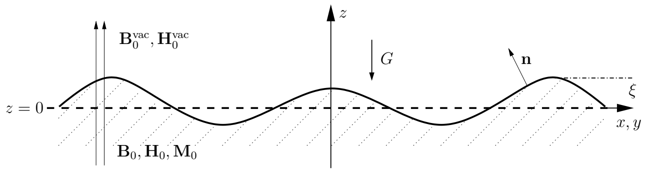

FIG. 1: Qualitative sketch of the geometry under consideration. The magnetic medium is occupying the negative half-space. The deflection of the deformable surface with respect to the flat surface at $z = 0$ is denoted by ξ with its unit normal vector $\mathbf{n}$ pointing upwards. The applied magnetic field is always parallel to the $z-$axis, while the acceleration due to gravity G is acting in the opposite direction.

component of a vector with the sum convention applied to repeated indices. Here ∂_t and ∂_i are partial derivatives with respect to time and space, respectively.

In our notation $\mathbf{v}$ is the velocity, p the pressure, $\mathbf{G}$ represents the acceleration due to gravity, $\mathbf{B}$ and $\mathbf{H}$ are the magnetic induction and the magnetic field, respectively, and the symmetric second rank tensors T_{ij} and ϵ_{ij} denote the stress tensor and the strain field, respectively. We do not make at this stage the incompressibility approximation in order to maintain the symmetric structure of the Navier-Stokes equation which turns out to be necessary for the adjoining process. Only at the end we will simplify the formulas by assuming incompressibility.

The underlying assumptions are as follows. Even though the magnetic field is considered a slowly relaxing variable in the hydrodynamic theory of Jarkova et al., we assume that it relaxes fast enough on the time scale considered in our discussion for the Rosensweig instability. This is justified by the fact, that the growth of surface spikes takes place on a time scale long compared to the temporal variations of the magnetic field. The magnetic field is then defined by the static Maxwell equations and the corresponding boundary conditions at the surface. We also assume, that the macroscopic material parameters like the shear modulus and the shear viscosity are independent of the magnetization in the medium. This also implies that we will neglect magnetostriction in our discussions.

As long as we consider the medium as compressible, we need an equation of state. Due to the assumption, that no magnetostrictive effects are important in our discussion, this equation of state can be assumed to be the barotropic equation

$$\delta p \;=\; c^2 \delta\rho \quad (4)$$

with the speed of sound c. As Jarkova et al. stated in[30], the modification of the speed of sound and especially the anisotropy effect in presence of an external magnetic field is proportional to the magnetostrictive constants and therefore of no importance in our discussion.

The stress tensor of the medium is defined via the conservation equation for the momentum density (1) and given in our notation by

$$
\begin{aligned}
T_{ij} &= g_i v_j + p\delta_{ij} - \big(B_i H_j - \frac{1}{2} B_k H_k\, \delta_{ij}\big) \\
&\quad -\mu_2(\epsilon_{jk}\epsilon_{ki} + \epsilon_{ik}\epsilon_{kj}) - \hat{\mu}\epsilon_{kk}\epsilon_{ij} - 2\mu_2\epsilon_{ij} \\
&\quad -\hat{\mu}\delta_{ij}\epsilon_{kk} - \nu_2(\partial_j v_i + \partial_i v_j) - \hat{\nu}\delta_{ij}\partial_k v_k \qquad (5)
\end{aligned}
$$

with the abbreviations $\hat{\mu} = \mu_1 - 2/3\,\mu_2$ and $\hat{\nu} = \nu_1 - 2/3\,\nu_2$ for the contributions vanishing in the limit of incompressibility, while the material parameters μ_2 and ν_2 stand for the shear elasticity and shear viscosity respectively. The compressional elasticity μ_1 and viscosity ν_1 are hidden in the abbreviations $\hat{\mu}$ and $\hat{\nu}$.

We consider the case of an infinitelly extended surface, initially situated at $z = 0$. For convenience the magnetic medium is filling the negative half-space whereas the vaccum is assumed to occupy the positive one; the gravitational force is assumed to point downwards. The applied magnetic field is oriented parallel to the $z-$axis (cf. Fig. 1).

To find the adjoint system of equations with its corresponding boundary conditions to the linear problem, we linearize Eqs. (1) to (3) with respect to the initially flat surface.

$$
\begin{aligned}
\{\rho, p, \mathbf{B}, \mathbf{H}, \mathbf{M}\} &= \{\rho_0, p_0, \mathbf{B}_0, \mathbf{H}_0, \mathbf{M}_0\} \\
&\quad + \{\rho^{(1)}, p^{(1)}, \mathbf{B}^{(1)}, \mathbf{H}^{(1)}, \mathbf{M}^{(1)}\} \qquad (6) \\
\{\mathbf{v}, \epsilon_{ij}, \xi\} &= 0 + \{\mathbf{v}^{(1)}, \epsilon_{ij}^{(1)}, \xi^{(1)}\} \qquad (7)
\end{aligned}
$$

We will drop, however, the superscript (1) in the following calculation. In our notation ξ describes the deflection of the surface from its initially flat state (cf. Fig. 1). The observables p_0, ρ_0, B_0, H_0 and M_0 are, respectively, the pressure, the density, the magnetic flux density, the magnetic field and the magnetization in the basic state where the surface is flat. They are related by the hydrostatic pressure relation.

IV. THE LINEAR EQUATIONS AND THE ADJOINT SYSTEM

When writing down the system of equations in linear order, we recall the fact already stated in[3] that the magnetic contributions cancel in the dynamic equations within the scope of the considered assumptions:

$$
\begin{aligned}
\rho_0\partial_t v_i + \partial_i p - \hat{\nu}\partial_i\partial_k v_k - \nu_2\partial_j(\partial_i v_j + \partial_j v_i) & \\
-\hat{\mu}\partial_i\epsilon_{kk} - 2\mu_2\partial_j\epsilon_{ij} &= 0 \qquad (8) \\
\partial_t\epsilon_{ij} - \frac{1}{2}(\partial_i v_j + \partial_j v_i) &= 0 \qquad (9) \\
\partial_t\rho + \rho_0\partial_i v_i &= 0 \qquad (10)
\end{aligned}
$$

The corresponding boundary conditions using the stress balance at the surface read in linear order

$$
\begin{aligned}
\mu_2\epsilon_{xz} + \nu_2(\partial_z v_x + \partial_x v_z) &= 0 \qquad (11) \\
\mu_2\epsilon_{yz} + \nu_2(\partial_z v_y + \partial_y v_z) &= 0 \qquad (12) \\
p - 2\mu_2\epsilon_{zz} - 2\nu_2\partial_z v_z - \hat{\mu}\epsilon_{ii} - \hat{\nu}\partial_i v_i & \\
-G\rho_0\xi + \frac{\mu_0}{1+\mu_0/\mu} M_0^2 k\xi - \sigma k^2\xi &= 0 \qquad (13)
\end{aligned}
$$

while for a free deformable surface we additionally have to fulfill the kinematic boundary condition

$$
v_z = \partial_t\xi \qquad (14)
$$

As stated already in Sec. II, the surface is subject to thermal fluctuations that we will expand in terms of plane waves with frequency ω and wave vector $\mathbf{k}$, $\xi = \hat{\xi} e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}$.

The system of dynamic equations in the medium can be written in terms of an eleven dimensional state vector, that we will define in the following way

$$
\Phi = \big(v_x, v_y, v_z, p, \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz}, \rho\big) \qquad (15)
$$

We will skip the discussion of the magnetic part of the system of equations. This part completely decouples from the dynamic part of the medium as stated above and reduces within our assumptions to the Laplace equation for the magnetic potential. The Laplace equation is self-adjoint and a homogenous equation. Therefore Fredholm's alternative is satisfied automatically.

Using the definition given above we can write the system of linear equations (8) to (9) together with the equation of state of the medium (4) in the following form, which can be taken to be the definition for the linear operator $\mathcal{L}_0$.

$$
\mathcal{L}_0\Phi = 0 \qquad (16)
$$

To find the adjoint operator $\mathcal{L}_0^\dagger$, and especially the adjoint boundary conditions, we use the following identity with $\bar{\Phi}$ denoting the adjoint state

$$
\langle\bar{\Phi} \mid \mathcal{L}_0\Phi\rangle = \langle\mathcal{L}_0^\dagger\bar{\Phi} \mid \Phi\rangle \qquad (17)
$$

The left hand side of this equation corresponds to the following integral using the standard scalar product, which we have to integrate by parts,

$$\lim_{L\to\infty}\frac{1}{4L^2}\int_{-L}^{L}dx\int_{-L}^{L}dy\int_{-\infty}^{\xi}dz\int_{0}^{t}dt\,\Big\{\bar{v}_x\Big\{\big(\rho\partial_t-\nu_2\partial_i^2-(\hat{\nu}+\nu)\partial_x^2\big)v_x-\hat{\nu}\partial_x\partial_y v_y-\nu_2\partial_y\partial_x v_y-\hat{\nu}\partial_x\partial_z v_z-\nu_2\partial_z\partial_x v_z$$
$$+\partial_x p-2\mu_2\partial_x\epsilon_{xx}-\hat{\mu}\partial_x\epsilon_{xx}-\hat{\mu}\partial_x\epsilon_{yy}-\hat{\mu}\partial_x\epsilon_{zz}-2\mu_2\partial_y\epsilon_{xy}-2\mu_2\partial_z\epsilon_{xz}\Big\}$$
$$+\bar{v}_y\Big\{-\hat{\nu}\partial_z\partial_x v_x-\nu_2\partial_x\partial_z v_x+\big(\rho\partial_t-\nu_2\partial_i^2-(\hat{\nu}+\nu)\partial_y^2\big)v_y-\hat{\nu}\partial_y\partial_z v_z-\nu_2\partial_z\partial_y v_z$$
$$+\partial_y p-\hat{\mu}\partial_y\epsilon_{xx}-2\mu_2\partial_y\epsilon_{yy}-\hat{\mu}\partial_y\epsilon_{yy}-\hat{\mu}\partial_y\epsilon_{zz}-2\mu_2\partial_x\epsilon_{xy}-2\mu_2\partial_z\epsilon_{yz}\Big\}$$
$$+\bar{v}_z\Big\{-\hat{\nu}\partial_z\partial_x v_x-\nu_2\partial_x\partial_z v_x-\hat{\nu}\partial_z\partial_y v_y-\nu_2\partial_y\partial_z v_y+\big(\rho\partial_t-\nu_2\partial_i^2-(\hat{\nu}+\nu)\partial_z^2\big)v_z$$
$$+\partial_z p-\hat{\mu}\partial_z\epsilon_{xx}-\hat{\mu}\partial_z\epsilon_{yy}-2\mu_2\partial_z\epsilon_{zz}-\hat{\mu}\partial_z\epsilon_{zz}-2\mu_2\partial_x\epsilon_{xz}-2\mu_2\partial_y\epsilon_{yz}\Big\}+\bar{p}\Big\{\partial_x v_x+\partial_y v_y+\partial_z v_z+\frac{\partial_t\rho}{\rho_0}\Big\}$$
$$+\bar{\epsilon}_{xx}\Big\{-\partial_x v_x+\partial_t\epsilon_{xx}\Big\}+\bar{\epsilon}_{yy}\Big\{-\partial_y v_y+\partial_t\epsilon_{yy}\Big\}+\bar{\epsilon}_{zz}\Big\{-\partial_z v_z+\partial_t\epsilon_{zz}\Big\}+\bar{\epsilon}_{xy}\Big\{-\frac{\partial_y}{2}v_x-\frac{\partial_x}{2}v_y+\partial_t\epsilon_{xy}\Big\}$$
$$+\bar{\epsilon}_{xz}\Big\{-\frac{\partial_z}{2}v_x-\frac{\partial_x}{2}v_z+\partial_t\epsilon_{xz}\Big\}+\bar{\epsilon}_{yz}\Big\{-\frac{\partial_z}{2}v_y-\frac{\partial_y}{2}v_z+\partial_t\epsilon_{yz}\Big\}+\bar{\rho}\Big\{\frac{\partial_t p}{\rho_0}-c^2\frac{\partial_t\rho}{\rho_0}\Big\}\Big\} \tag{18}$$

This leads to the adjoint linear operator $\mathcal{L}_0^\dagger=\begin{pmatrix}A & C\\ B & D\end{pmatrix}$ with the abbreviations (19)

$$A=\begin{pmatrix}-\rho\partial_t-\nu_2\partial_i^2-(\hat{\nu}+\nu_2)\partial_x^2 & -\hat{\nu}\partial_x\partial_y-\nu_2\partial_y\partial_x & -\hat{\nu}\partial_x\partial_z-\nu_2\partial_z\partial_x & -\partial_x+\frac{1}{\rho_0}(\partial_x\rho_0)\\ -\hat{\nu}\partial_y\partial_x-\nu_2\partial_x\partial_y & -\rho\partial_t-\nu_2\partial_i^2-(\hat{\nu}+\nu_2)\partial_y^2 & -\hat{\nu}\partial_y\partial_z-\nu_2\partial_z\partial_y & -\partial_y+\frac{1}{\rho_0}(\partial_y\rho_0)\\ -\hat{\nu}\partial_z\partial_x-\nu_2\partial_x\partial_z & -\hat{\nu}\partial_z\partial_y-\nu_2\partial_y\partial_z & -\rho\partial_t-\nu_2\partial_i^2-(\hat{\nu}+\nu_2)\partial_z^2 & -\partial_z+\frac{1}{\rho_0}(\partial_z\rho_0)\\ -\partial_x & -\partial_y & -\partial_z & 0\end{pmatrix} \tag{20}$$

$$B=\begin{pmatrix}2\mu_2\partial_x+\hat{\mu}\partial_x & \hat{\mu}\partial_y & \hat{\mu}\partial_z & 0\\ \hat{\mu}\partial_x & 2\mu_2\partial_y+\hat{\mu}\partial_y & \hat{\mu}\partial_z & 0\\ \hat{\mu}\partial_x & \hat{\mu}\partial_y & 2\mu_2\partial_z+\hat{\mu}\partial_z & 0\\ 2\mu_2\partial_y & 2\mu_2\partial_x & 0 & 0\\ 2\mu_2\partial_z & 0 & 2\mu_2\partial_x & 0\\ 0 & 2\mu_2\partial_z & 2\mu_2\partial_y & 0\\ 0 & 0 & 0 & -\frac{1}{\rho_0}\partial_t\end{pmatrix} \tag{21}$$

$$C=\begin{pmatrix}\partial_x & 0 & 0 & \frac{1}{2}\partial_y & \frac{1}{2}\partial_z & 0 & 0\\ 0 & \partial_y & 0 & \frac{1}{2}\partial_x & 0 & \frac{1}{2}\partial_y & 0\\ 0 & 0 & \partial_z & 0 & \frac{1}{2}\partial_x & \frac{1}{2}\partial_y & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\rho_0}\partial_t\end{pmatrix} \tag{22}$$

$$D=-\begin{pmatrix}\partial_t & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \partial_t & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \partial_t & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \partial_t & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \partial_t & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \partial_t & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{c^2}{\rho_0}\partial_t\end{pmatrix} \tag{23}$$

While integrating eq. (18) by parts, one also obtains surface contributions, which have to vanish to fulfill eq. (17). The most important parts are the contributions due to the $z-$integration. At the bottom ($z=-\infty$) they are always 0, since the eigenvectors of the linear system exponentially decay with increasing depth. The condition, that they should also vanish at the surface, defines the adjoint boundary conditions at the free surface. At this point we can just state that the following sum should vanish

$$\bar{v}_x(-\nu_2\partial_z v_x-\nu_2\partial_x v_z-\mu_2\epsilon_{xz})+v_x(\nu_2\partial_z\bar{v}_x+\nu_2\partial_x\bar{v}_z-\tfrac{1}{2}\bar{\epsilon}_{xz})+\bar{v}_y(-\nu_2\partial_z v_y-\nu_2\partial_y v_z-\mu_2\epsilon_{yz})+v_y(\nu_2\partial_z\bar{v}_y+\nu_2\partial_y\bar{v}_z-\tfrac{1}{2}\bar{\epsilon}_{yz})+\bar{v}_z(-\hat{\nu}\partial_i v_i-2\nu_2\partial_z v_z-\hat{\mu}\epsilon_{ii}+p-2\mu_2\epsilon_{zz})+v_z(\hat{\nu}\partial_i\bar{v}_i+2\nu_2\partial_z\bar{v}_z+\bar{p}-\bar{\epsilon}_{zz})=0 \tag{24}$$

Using the two tangential boundary conditions of the original system (11,12) those contributions in (24) vanish that are proportional to $\bar{v}_x$ or $\bar{v}_y$. Using the normal stress boundary condition (13) in the second last term of (24), we implement the gravitational, the surface tension and the magnetic contributions into the adjoint boundary conditions. This also ensures the presence of the driving force in the boundary conditions of the adjoint system. With the help of the kinematic boundary condition in the original case (14), which reduces to $v_z = i\omega\xi$, we can then substitute v_z ending up with the necessary condition at the surface

$$v_x(\nu_2\partial_z\bar{v}_x + \nu_2\partial_x\bar{v}_z - \frac{1}{2}\bar{\epsilon}_{xz}) + v_y(\nu_2\partial_z\bar{v}_y + \nu_2\partial_y\bar{v}_z - \frac{1}{2}\bar{\epsilon}_{yz}) + \xi\Big(G\rho_0\bar{v}_z - \frac{\mu_0 M_0^2}{1+\mu_0/\mu}k\bar{v}_z + \sigma k^2\bar{v}_z + i\omega\bar{p} + i\omega\hat{\nu}\partial_i\bar{v}_i + 2i\omega\nu_2\partial_z\bar{v}_z - i\omega\bar{\epsilon}_{zz}\Big) = 0 \quad (25)$$

We can split this condition into three separate parts. This choice is suggested by the fact, that within the scalar product we used, the velocities v_x and v_y are independent components. We therefore find as boundary conditions at the free surface in the adjoint case

$$\nu_2\partial_z\bar{v}_x + \nu_2\partial_x\bar{v}_z - 1/2\,\bar{\epsilon}_{xz} = 0 \quad (26)$$
$$\nu_2\partial_z\bar{v}_y + \nu_2\partial_y\bar{v}_z - 1/2\,\bar{\epsilon}_{yz} = 0 \quad (27)$$
$$G\rho_0\bar{v}_z - \frac{\mu_0 M_0^2}{1+\mu_0/\mu}k\bar{v}_z + \sigma k^2\bar{v}_z + i\omega\bar{p} + i\omega\hat{\nu}\partial_k\bar{v}_k + 2i\omega\nu_2\partial_z\bar{v}_z - i\omega\bar{\epsilon}_{zz} = 0 \quad (28)$$

The horizontal boundary conditions originating from the horizontal integrations are satisfied automatically, since we take the limit of an infinitely extended layer. The only additional condition we get is due to the time integration, but this is not important in the limit of a stationary instability we will discuss in the following, but it should be taken into account if one handles oscillatory instabilities, e.g. the Faraday instability.

At this point we restrict our calculations to an incompressible medium assuming $\partial_i\bar{v}_i = 0 = \bar{\epsilon}_{ii}$. The adjoint system of equations, $\mathcal{L}_0^\dagger\bar{\Phi} = 0$, then reads

$$-\rho\partial_t\bar{v}_i - \partial_i\bar{p} - \nu_2\partial_j\partial_j\bar{v}_i + \frac{1}{2}(\partial_i\bar{\epsilon}_{\underline{ii}} + \partial_j\epsilon_{ij}) = 0 \quad (29)$$
$$-\partial_t\bar{\epsilon}_{ij} + 2\mu_2(\partial_i\bar{v}_j + \partial_j\bar{v}_i)(1 - \frac{1}{2}\delta_{\underline{ij}}) = 0 \quad (30)$$
$$\partial_i\bar{v}_i = 0 \quad (31)$$

where underlined indices are not summed upon. Their structure is similar to those of the original equations.

Following the same approach as for the original linear system, namely using the dynamic equations for the strain field in the momentum conservation equation, we get as a first step

$$\rho_0\bar{\omega}^2\bar{v}_i - i\bar{\omega}\partial_i\bar{p} + \mu(\bar{\omega})\partial_j\left(\partial_j\bar{v}_i + \partial_i\bar{v}_j\right) = 0 \quad (32)$$
$$\partial_i\bar{v}_i = 0 \quad (33)$$

where we used the abbreviation $\mu(\bar{\omega}) = \mu_2 - i\bar{\omega}\nu_2$. We will separate the velocity field into two parts. One due to potential flow and the second due to vorticity flow. Fulfilling the dynamic bulk equations, we obtain the inverse decay length $\bar{q}^2 = k^2 - \rho\bar{\omega}^2/\mu(\bar{\omega})$ for the vorticity flow with respect to the $z-$axis. The solvability condition for the adjoint boundary conditions then leads to the dispersion relation for surface waves in the adjoint system.

$$\bar{\omega}^2\rho\left(2k^2\mu(\bar{\omega}) - \bar{\omega}^2\rho\right) + \bar{\omega}^2\rho\left[-\frac{\bar{\omega}}{\omega}\left(G\rho - \frac{\mu_0 M_0^2}{1+\mu_0/\mu}k + \sigma k^2\right)k + 2\mu(\bar{\omega})k^2\right] - (2k^2\mu(\bar{\omega}))^2\left(1 - \sqrt{1 - \frac{\bar{\omega}^2\rho}{\mu(\bar{\omega})k^2}}\right) = 0 \quad (34)$$

Eq. (34) reduces to the original dispersion relation[3] if $\bar{\omega} = -\omega$. This is the physical solution, since the adjoint space acquires an easy and obvious physical interpretation: Considering the surface in its general form with left and right traveling waves and the corresponding adjoint surface deflection using the solution given above

$$\xi = \hat{\xi}_R e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}} + \hat{\xi}_L e^{-i\omega t - i\mathbf{k}\cdot\mathbf{r}} + c.c. \quad (35)$$
$$\bar{\xi} \equiv \hat{\bar{\xi}}_R e^{i\bar{\omega} t - i\mathbf{k}\cdot\mathbf{r}} + \hat{\bar{\xi}}_L e^{-i\bar{\omega} t - i\mathbf{k}\cdot\mathbf{r}} + c.c. = \hat{\bar{\xi}}_R e^{-i\omega t - i\mathbf{k}\cdot\mathbf{r}} + \hat{\bar{\xi}}_L e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}} + c.c. \quad (36)$$

(*c.c.* is the complex conjugate) we recognize by comparing equations (35) and (36), that a right traveling wave transforms into a left traveling wave in the adjoint system and vice versa, leading to the conditions:

$$\hat{\bar{\xi}}_R = \hat{\xi}_L \quad \text{and} \quad \hat{\bar{\xi}}_L = \hat{\xi}_R \quad (37)$$

V. ADJOINT EIGENVECTORS FOR THE ROSENSWEIG INSTABILITY

Up to now all calculations have been performed without giving an explicit expression for the kinematic boundary condition at the surface in the adjoint system. To calculate the adjoint eigenvectors we have to specify that condition. However, it cannot be obtained by integrating by parts, since it is of a completely different type compared to the boundary conditions (11) to (13). While the former ones are derived using the stress balance at the surface, the kinematic boundary condition is phenomenological in nature. For surface waves in the adjoint space we therefore require a kinematic boundary condition of exactly the same structure as in the original case.

$$v_z(\omega) = \partial_t\xi(\omega) = i\omega\xi(\omega) \quad (38)$$
$$\bar{v}_z(\bar{\omega}) = \partial_t\bar{\xi}(\bar{\omega}) = i\bar{\omega}\bar{\xi}(\bar{\omega}) \quad (39)$$

Following the same way to calculate the eigenvectors as in the case of the original system[9], we get for the amplitudes of the vorticity flow potential (recall $\bar{q}^2 = k^2 - \rho\bar{\omega}^2/(\mu_2 - i\bar{\omega}\nu_2)$)

$$\bar{\tilde{\Psi}}_x = ik_y \frac{2k}{\bar{q}^2 + k^2}\bar{\tilde{\varphi}} \tag{40}$$

$$\bar{\tilde{\Psi}}_y = -ik_x \frac{2k}{\bar{q}^2 + k^2}\bar{\tilde{\varphi}} \tag{41}$$

and of the scalar potential

$$\bar{\tilde{\varphi}} = i\bar{\omega}\frac{\bar{q}^2 + k^2}{k(\bar{q}^2 - k^2)} \tag{42}$$

where $\bar{\Psi}_i$ denote the components of the vector potential of the velocity defined by the rotational part of flow $\bar{\mathbf{v}}^{\text{rot}} = \nabla \times \bar{\mathbf{\Psi}}$. The scalar potential $\bar{\varphi}$ is connected to the potential flow $\bar{\mathbf{v}}^{\text{pot}} = \nabla\bar{\varphi}$.

The components of the adjoint velocities then become similar to the ones known from the original system, and – as in the original system – they vanish in the case of a stationary instability

$$\bar{v}_x = \bar{\omega}\frac{k_x}{k}\left(e^{kz} - \frac{2\bar{q}k}{\bar{q}^2 + k^2}e^{\bar{q}z}\right)\frac{\bar{q}^2 + k^2}{\bar{q}^2 - k^2}\bar{\xi} \tag{43}$$

$$\bar{v}_y = \bar{\omega}\frac{k_y}{k}\left(e^{kz} - \frac{2\bar{q}k}{\bar{q}^2 + k^2}e^{\bar{q}z}\right)\frac{\bar{q}^2 + k^2}{\bar{q}^2 - k^2}\bar{\xi} \tag{44}$$

$$\bar{v}_z = i\bar{\omega}\left(e^{kz} - \frac{2k^2}{\bar{q}^2 + k^2}e^{\bar{q}z}\right)\frac{\bar{q}^2 + k^2}{\bar{q}^2 - k^2}\bar{\xi} \tag{45}$$

For the adjoint strain field we get

$$\bar{\epsilon}_{zz} = 2\mu_2 k\left(e^{kz} - \frac{2\bar{q}k}{\bar{q}^2 + k^2}e^{\bar{q}z}\right)\frac{\bar{q}^2 + k^2}{\bar{q}^2 - k^2}\bar{\xi} \tag{46}$$

$$\bar{\epsilon}_{xx} = 2\mu_2\frac{k_x^2}{k}\left(e^{kz} - \frac{2\bar{q}k}{\bar{q}^2 + k^2}e^{\bar{q}z}\right)\frac{\bar{q}^2 + k^2}{\bar{q}^2 - k^2}\bar{\xi} \tag{47}$$

$$\bar{\epsilon}_{yy} = 2\mu_2\frac{k_y^2}{k}\left(e^{kz} - \frac{2\bar{q}k}{\bar{q}^2 + k^2}e^{\bar{q}z}\right)\frac{\bar{q}^2 + k^2}{\bar{q}^2 - k^2}\bar{\xi} \tag{48}$$

$$\bar{\epsilon}_{xy} = -4\mu_2\frac{k_x k_y}{k}\left(e^{kz} - \frac{2\bar{q}k}{\bar{q}^2 + k^2}e^{\bar{q}z}\right)\frac{\bar{q}^2 + k^2}{\bar{q}^2 - k^2}\bar{\xi} \tag{49}$$

$$\bar{\epsilon}_{xz} = -4i\mu_2 k_x\left(e^{kz} - e^{\bar{q}z}\right)\frac{\bar{q}^2 + k^2}{\bar{q}^2 - k^2}\bar{\xi} \tag{50}$$

$$\bar{\epsilon}_{yz} = -4i\mu_2 k_y\left(e^{kz} - e^{\bar{q}z}\right)\frac{\bar{q}^2 + k^2}{\bar{q}^2 - k^2}\bar{\xi} \tag{51}$$

Obviously the adjoint strain components have the same structure as the components in the original case and they also show a finite stationary limit. However, they do not have the same units. While the strain field in the original case is dimensionless, the adjoint strain field is proportional to the shear modulus μ_2. This is consistent with the scalar product (18), where all contributions need to have the same dimension. One could avoid the dimension of the adjoint strain field by defining a scalar product with a metric containing units in the fifth to the tenth component.

The reasons why previous attempts to solve the adjoint problem have failed are, in our opinion, twofold. One crucial part in our discussion is to treat the medium as compressible.

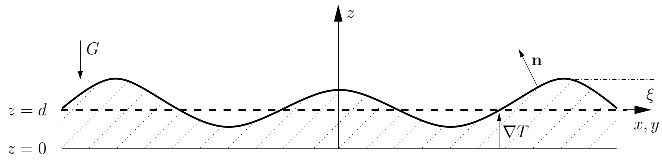

FIG. 2: Qualitative sketch of the geometry under consideration in the case of pure Marangoni convection. The fluid is confined between the rigid surface at $z = 0$ and the deformable surface initially at $z = d$. The deflection of the deformable surface with respect to the flat surface is denoted by ξ with its unit normal vector $\mathbf{n}$ pointing upwards. The applied temperature gradient is always parallel, the acceleration due to gravity G always antiparallel to the z axis.

This ensures e.g. the presence of the contribution $\sim \partial_j\partial_i v_j$ in the Navier-Stokes equation. During the process of adjoining, commutativity of gradients in this term requires that the surface terms $\sim \bar{v}_i\partial_i v_j$ and $\sim \bar{v}_j\partial_i v_i$ are equivalent, which would be violated if incompressibility is applied before. The assumption of an incompressible fluid is therefore too strong a restriction. An even more important point is to treat the system as a dynamic one. The subtle reason for that is manifest in the dynamic boundary condition of the surface deflection. Assuming stationarity from the beginning would imply an always undeformed surface because the vertical velocity at the surface would vanish in any case. However, this velocity component needs to be finite to allow the surface to deform. The marginal point where the spikes are about to develop (or the final point where the spikes have fully developed) are then obtained as the static limit $\omega \to 0$ of the full dynamic behavior.

VI. THE ADJOINT PROBLEM FOR THE MARANGONI INSTABILITY

Inspired by the result in the case of the Rosensweig instability, we apply the same formalism to the case of stationary Marangoni convection to find the adjoint system of equations for this case as well. However, there exists a crucial difference between these two instabilities. While in the case of magnetic fluids the external force acts normal to the free surface, in the case of Marangoni convection the external force is acting tangential to the surface (see Fig. 2). This external force for the Marangoni instability is mediated by temperature fluctuations. The surface tension σ is therefore assumed to be temperature dependent and reads in a series expansion up to linear order in T

$$\sigma(T) = \sigma(T_R) - \gamma(T - T_R) \tag{52}$$

with the change in surface tension due to temperature fluctuations $\gamma = -(\partial\sigma(T)/\partial T)_{T=T_R}$ and where T_R represents an arbitrary reference temperature. For the following discussion we will refer to $\sigma(T_R)$ as σ.

A. Basic equations and the adjoint system

To find the adjoint system for the purely surface driven convection, the Marangoni instability, we assume a viscous Newtonian fluid. As done in the case of the Rosensweig instability, we assume it to be compressible with a barotropic equation of state at the beginning, but in the end we will again use the limit of an incompressible fluid. Additionally we have to incorporate the equation of heat transport with the temperature T and the thermal diffusivity χ. All the other variables are denoted in the same way as in the previous discussion. As we want to discuss the purely surface driven contribution of convection, all contributions due to buoyancy are neglected. The system of equations thus reads

$$\partial_t \rho + \partial_k(\rho v_k) = 0 \tag{53}$$
$$\partial_t g_i + \partial_j T_{ij} = \rho G_i \tag{54}$$
$$\partial_t T + v_j \partial_j T = \chi \partial_j \partial_j T \tag{55}$$

The stress tensor T_{ij} of the fluid under consideration takes the form

$$T_{ij} = v_j g_i + p\delta_{ij} - \nu_2(\partial_j v_i + \partial_i v_j) - \hat{\nu}(\partial_k v_k)\delta_{ij} \tag{56}$$

We require the normal as well as the tangential stress at the free surface between the Newtonian fluid and the vacuum to be balanced, leading to the normal and tangential boundary conditions, respectively

$$p - \rho_0 G\xi - 2\nu_2 \partial_z v_z - \hat{\nu}(\partial_k v_k) = -\sigma(\partial_x^2 + \partial_y^2)\xi \tag{57}$$
$$\nu_2(\partial_y v_z + \partial_z v_y) = -\gamma \partial_y T + \gamma\beta\partial_y \xi \tag{58}$$
$$\nu_2(\partial_x v_z + \partial_z v_x) = -\gamma \partial_x T + \gamma\beta\partial_x \xi \tag{59}$$

where β denotes the applied temperature gradient across the fluid. Additionally we have to specify the phenomenological boundary conditions at the surface. Again the kinematic boundary condition (14) for a free deformable surface is assumed to hold. Second, we assume the heat flux Q through the surface to be proportional to the local temperature gradient, where κ denotes the coefficient of (surface) heat conduction.

$$Q(T) = -\kappa \partial_z T \tag{60}$$

At the bottom ($z = 0$) of the container we assume the usual rigid boundary conditions

$$v_i = \partial_z v_z = T = 0 \tag{61}$$

The state vector now becomes six dimensional and is defined by

$$\Phi = (v_x, v_y, v_z, p, T, \rho) \tag{62}$$

so that the system of equations reads again in the general form

$$\mathcal{L}_0 \Phi = 0 \tag{63}$$

We use the usual scalar product, however, now the $z-$integration is bounded between the bottom plate ($z = 0$) and the free surface ($z = \xi$).

$$\langle \bar{\Phi} \mid \Phi \rangle = \lim_{L\to\infty} \frac{1}{4L} \int_{-L}^{L} dx \int_{-L}^{L} dy \int_{0}^{\xi} dz \int_{0}^{t} dt \; \bar{\Phi}\Phi \tag{64}$$

The adjoint linear operator then turns out to be

$$\begin{pmatrix} -\rho\partial_t - \nu_2\partial_i^2 - (\hat{\nu}+\nu_2)\partial_x^2 & -\hat{\nu}\partial_x\partial_y - \nu_2\partial_y\partial_x & -\hat{\nu}\partial_x\partial_z - \nu_2\partial_z\partial_x & -\partial_x & 0 & 0 \\ -\hat{\nu}\partial_y\partial_x - \nu_2\partial_x\partial_y & -\rho\partial_t - \nu_2\partial_i^2 - (\hat{\nu}+\nu_2)\partial_y^2 & -\hat{\nu}\partial_y\partial_z - \nu_2\partial_z\partial_y & -\partial_y & 0 & 0 \\ -\hat{\nu}\partial_z\partial_x - \nu_2\partial_x\partial_z & -\hat{\nu}\partial_z\partial_y - \nu_2\partial_y\partial_z & -\rho\partial_t - \nu_2\partial_i^2 - (\hat{\nu}+\nu_2)\partial_z^2 & -\partial_z & -\beta & 0 \\ -\partial_x & -\partial_y & -\partial_z & 0 & 0 & -\frac{1}{\rho_0}\partial_t \\ 0 & 0 & 0 & 0 & -\partial_t - \chi\partial_i^2 & 0 \\ 0 & 0 & 0 & -\frac{1}{\rho_0}\partial_t & 0 & -\frac{c^2}{\rho_0}\partial_t \end{pmatrix}$$

The surface contributions of the integration by parts should vanish to fulfill eq. (17) leading to the corresponding boundary conditions in the adjoint case.

$$2i\omega\nu_2\partial_z\bar{v}_z + i\omega\hat{\nu}(\partial_k\bar{v}_k) + i\omega\bar{p} + \rho G\bar{v}_z + \sigma k^2\bar{v}_z = 0 \tag{65}$$
$$\bar{v}_x(-ik_x)\hat{T}(z) - \bar{v}_x\gamma\beta(-ik_x) + \hat{v}_x(z)\nu_2(\partial_z\bar{v}_x + \partial_x\bar{v}_z) = 0 \tag{66}$$
$$\bar{v}_y(-ik_y)\hat{T}(z) - \bar{v}_y\gamma\beta(-ik_y) + \hat{v}_y(z)\nu_2(\partial_z\bar{v}_y + \partial_y\bar{v}_z) = 0 \tag{67}$$
$$-\chi\bar{T}\partial_z T + \chi T\partial_z\bar{T} = 0 \tag{68}$$

In the last set of equations we have used the fact, that every variable of the original system is modulated by ξ, in particular we used $T(z) = \hat{T}(z)\xi$ and $v_{x,y}(z) = \hat{v}_{x,y}(z)\xi$. Actually Eq. (68) just states, that the adjoint temperature may differ from the original one by just a constant. For the phenomenological boundary conditions we take the same form as for the original case, namely

$$\bar{v}_z = i\bar{\omega}\bar{\xi} \tag{69}$$
$$\bar{Q}(\bar{T}) = -\kappa\,\partial_z\bar{T} \tag{70}$$

The boundary conditions at the rigid bottom turn out to be self-adjoint, but are repeated here

$$\bar{v}_i = \partial_z \bar{v}_z = 0 \quad (71)$$
$$\bar{T} = 0 \quad (72)$$

B. The dimensionless representation

For the further discussion we give the dimensionless version of the problem discussed in the previous section, because it is common in all the other discussion regarding convection. Following the usual steps[31], the linearized dynamical equations for the deviations from the conducting state of the temperature θ and the vertical component of the velocity v_z read

$$(D^2 - k^2)(D^2 - k^2 - i\omega)v_z(z) = 0 \quad (73)$$
$$(D^2 - k^2 - i\omega\mathcal{P})\theta(z) = -v_z(z) \quad (74)$$

The boundary conditions at the free surface using the stress balance then read

$$(D^2 + k^2)v_z(z) = -\mathcal{M}k^2\left(\theta(z) - \frac{1}{\mathcal{P}}\xi\right) \quad (75)$$
$$\mathcal{C}\mathcal{P}(i\omega - D^2 + 3k^2)Dv_z(z) = -(\mathcal{B} - k^2)k^2\xi \quad (76)$$

And for the phenomenological boundary conditions we gain

$$v_z(z) = i\omega\xi \quad (77)$$
$$\mathcal{P}(D + \mathcal{F})\theta(z) = \mathcal{F}\xi \quad (78)$$

At the bottom, the equations reduce to

$$v_z = Dv_z = \theta = 0 \quad (79)$$

While rescaling the variables we have introduced dimensionless numbers such as the Prandtl number $\mathcal{P} = \nu_2/\chi$, the Marangoni number $\mathcal{M} = \gamma\beta d^2/(\rho\chi\nu_2)$, the Crispation number $\mathcal{C} = \rho\nu_2\chi/(\sigma d)$, the Bond number $\mathcal{B} = \rho G d^2/\sigma$ and the Biot number $\mathcal{F} = (\partial Q/\partial T)d/\kappa$ as well as the dimensionless derivative with respect to z, $D = d/dz$.

Using the same arguments with the adjoint set of equations we find

$$(D^2 - k^2)(D^2 - k^2 + i\bar{\omega})\bar{v}_z(z) = -A\bar{\theta}(z) \quad (80)$$
$$(D^2 - k^2 + i\bar{\omega}\mathcal{P})\bar{\theta}(z) = 0 \quad (81)$$

It is worth mentioning here, that in eq. (80) an additional number, $A = \beta^2 d^4/(\chi\nu_2)$, arises. This is, however, consistent with condition (68), which allows the temperature in the adjoint case to differ from the original temperature by a constant factor. One could rescale the dimensionless adjoint temperature by exactly this number A, resulting in a dimensionalized adjoint temperature. This, however, is not surprising since also in the discussion of the adjoint system of the Rosensweig problem, the adjoint strain field acquired a different physical unit due to the dynamic coupling between velocity field and the strain field. The adjoint boundary conditions stemming from the adjoining process turn out to be

$$-\mathcal{M}(D\bar{v}_z(z))k^2\left(\hat{\theta} - \frac{1}{\mathcal{P}}\right) = (D\hat{v}_z)(D^2+k^2)\bar{v}_z(z) \quad (82)$$
$$\mathcal{C}\mathcal{P}(\omega\bar{\omega} - i\omega D^2 + 3i\omega k^2)D\bar{v}_z(z) = -(\mathcal{B} - k^2)k^2\bar{v}_z(z) \quad (83)$$

While the ones describing the free surface are

$$\bar{v}_z(z) = i\bar{\omega}\bar{\xi} \quad (84)$$
$$\mathcal{P}(D + \mathcal{F})\bar{\theta}(z) = \mathcal{F}\bar{\xi} \quad (85)$$

The self-adjoint boundary conditions at the bottom are repeated here in dimensionless form

$$\bar{v}_z = D\bar{v}_z = 0 \quad (86)$$
$$\bar{\theta} = 0 \quad (87)$$

In the dimensionless representation we explicitly made use of the fact that the macroscopic variables are modulated by ξ, in particular we used $Dv_z(z) = (D\hat{v}_z(z))\xi$ and $\theta(z) = \hat{\theta}(z)\xi$.

At that point we should mention a crucial point. While the adjoint boundary conditions in the case of the Rosensweig instability (26)-(28) turned out to be independent of the eigenvectors of the original case, the tangential boundary condition (82) contains the eigenvectors of the original case. By inspection of the adjoining-process this is due to coupling between the temperature and the velocity field, even though this coupling does not drive the instability. A similar coupling in the bulk equations of the Rosensweig case – the magnetic field to the velocity or the strain field – was missing. As a consequence, the adjoint dispersion relation will also depend on the original eigenvectors, which is discussed in detail in the Appendix.

C. The dispersion relation

We start solving the system of equations in the original case. Previous analytical work accounting for a stationary instability with finite deformation of the surface always assumed stationary equations from the beginning. However, to find a connection between the adjoint and original case, we need the general dispersion relation of surface waves propagating on the free surface.

To solve the dynamical equations (73) and (74) subject to the boundary conditions (75)-(79) we used an ansatz with hyperbolic functions[32]. In particular we used, after substitution of eq. (74) into eq. (73), the following solutions

$$\theta(z) = \sum_{i=1}^{3}\big(A_i\cosh(\lambda_i z) + B_i\sinh(\lambda_i z)\big) \quad (88)$$
$$v_z(z) = -\sum_{i=1}^{2}\big(\lambda_i^2 - k^2 - i\omega\mathcal{P}\big) \times\big(A_i\cosh(\lambda_i z) + B_i\sinh(\lambda_i z)\big) \quad (89)$$

together with the roots

$$\lambda_1^2 = k^2 \quad (90)$$
$$\lambda_2^2 = k^2 + i\omega \quad (91)$$
$$\lambda_3^2 = k^2 + i\omega\mathcal{P} \quad (92)$$

The solvability condition of the boundary conditions gives the corresponding dispersion relation of plane waves traveling on the surface of the fluid. However, the dispersion relation can only be given implicitly and is shown in the Appendix

$$\mathcal{D}(\omega, k, \mathcal{M}) \;=\; 0 \tag{93}$$

We will restrict ourselves in this discussion to the stationary case, although solutions of eq. (93) with a finite frequency ω might exist at the threshold. On the other hand one can prove analytically, that a nontrivial solution of eq. (93) is $\omega = 0$. Using this result we can perform the limit of a stationary instability and the solvability condition in the stationary case reduces to the neutral curve

$$\mathcal{M} \;=\; \frac{8k(\mathcal{B}{+}k^2)(k\cosh(k){+}\mathcal{F}\sinh(k))(2k{-}\sinh(2k))}{8\mathcal{C}k^5\cosh k + (\mathcal{B}{+}k^2)\big(\sinh^3(k){-}k^3\cosh(k)\big)} \tag{94}$$

which coincides with the result obtained by Takashima[22] assuming stationarity from the beginning. In the limit of vanishing surface deformations ($\mathcal{C} \to 0$) we find the same results as Pearson[18], Nield[19] as a special case.

These calculations and also the following ones have been checked using the ansatz of Nield[19], who used Fourier modes.

D. The adjoint dispersion relation

As in the the case of the Rosensweig instability, to get the adjoint system, one has to start with the fully dynamic problem. Using again hyperbolic functions the solutions can be written as

$$\bar{v}_z(z) \;=\; \sum_{i=1}^{3}\Big(\bar{A}_i\cosh(\bar{\lambda}_i z) + \bar{B}_i\sinh(\bar{\lambda}_i z)\Big) \tag{95}$$

$$\bar{\theta}(z) \;=\; -(i\bar{\omega} - k^2 + \bar{\lambda}_3)(\bar{\lambda}_3 - k^2) \times\Big(\frac{\bar{A}_3}{A}\cosh(\bar{\lambda}_3 z) + \frac{\bar{B}_3}{A}\sinh(\bar{\lambda}_3 z)\Big) \tag{96}$$

together with the adjoint roots

$$\bar{\lambda}_1^2 \;=\; k^2 \tag{97}$$

$$\bar{\lambda}_2^2 \;=\; k^2 - i\bar{\omega} \tag{98}$$

$$\bar{\lambda}_3^2 \;=\; k^2 - i\bar{\omega}\mathcal{P} \tag{99}$$

With the help of the adjoint boundary conditions, we obtain the dispersion relation of surface waves in the adjoint space, that can only be given implicitly again (see Appendix)

$$\bar{\mathcal{D}}(\bar{\omega}, \omega, k, \mathcal{M}) \;=\; 0 \tag{100}$$

This equation also gives $\bar{\omega}$ as a function of the frequency in the original case ω, although the expression is more complicated than for the case of the Rosensweig instability and a solution of Eq. (100) has not been obtained analytically. Nevertheless we have to guarantee that eq. (100) is fulfilled even when approaching the critical point for the stationary instability. When expanding eq. (100) in terms of ω we obtain

$$\bar{\mathcal{D}}(\bar{\omega}, \omega, k, \mathcal{M}) \;=\; \bar{\mathcal{D}}_0(\bar{\omega}, \omega{=}0, k, \mathcal{M}) + \;\bar{\mathcal{D}}_1(\bar{\omega}, \omega{=}0, k, \mathcal{M})\,\omega + \mathcal{O}(\omega^2) \tag{101}$$

When approaching the marginal point, $\bar{\mathcal{D}}_1$ and all the contributions of higher order in ω cancel with ω becoming 0. To fulfill eq. (101), additionally $\bar{\mathcal{D}}_0$ has to vanish. It can be shown, that if $\bar{\omega}$ as a function ω vanishes when ω vanishes, the constant contribution $\bar{\mathcal{D}}_0$ becomes zero and the adjoint dispersion relation is satisfied (see Appendix). Therefore the instability in the adjoint case occurs at the same point with the same characteristics.

VII. SUMMARY AND OUTLOOK

In this article we present a method to find the adjoint system of equations and the corresponding boundary conditions for surface driven instabilities with a deformable surface. In particular we discuss explicitly the case of the Rosensweig instability (in magnetic gels) and the case of pure Marangoni convection. As a special case the adjoint system for the case of pure ferrofluids is obtained straightforwardly when taking the limit of a vanishing shear modulus.

For the adjoining process it turns out to be very important, that the system is treated as a dynamical one and as compressible. In the end, however, the general set of adjoint equations can be simplified using the incompressibility condition.

The relation between the adjoint and the original frequency in the case of the Rosensweig instability is simply $\bar{\omega} = -\omega$. This solution has a straightforward physical interpretation, which is that left traveling waves in the original space transform into right traveling waves in the adjoint space and vice versa. The expressions for the adjoint eigenvectors take the same structure as the ones in the original case, but due to the dynamic coupling between the strain field and the velocity field, the adjoint strain field acquires units of a shear modulus. In the case of Marangoni convection the relation between the original and the adjoint frequency is more complicated and is only given implicitly. Nevertheless, the adjoint dispersion relation for this case is also fulfilled in the stationary limit.

The obtained results now allow a weakly nonlinear analysis of instabilities with a deformable surface based on the fundamental hydrodynamic equations leading to the desired amplitude equation. In the case of the Rosensweig instability this has already been done and will be presented in a future article.

Acknowledgments

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Appendix: Dispersion relations for Marangoni convection

In this Appendix we give the dispersion relations of the original and the adjoint Marangoni problem. In particular we discuss the adjoint dispersion relation in the limit of $\omega \to 0$.

The solvability condition of the system of dynamic equations (53 -56) together with the boundary conditions (57 -61) at the deformable surface yields the dispersion relation. It describes the relation between the frequency and the wave vector of surface waves propagating on a free surface. In an implicit form (and using $\lambda_1 = k$) it reads

$$\begin{aligned}
\mathcal{D}(\omega,k,\mathcal{M}) \equiv\ & i\omega^5\mathcal{P}^3(\mathcal{P}-1)k\Big\{k\Big[i\omega\mathcal{P}\mathcal{C}\lambda_3\cosh(\lambda_3)\Big(2k\lambda_2\big(k^2(4i\omega\mathcal{P}(\mathcal{P}-1)+\mathcal{M})-2\omega^2\mathcal{P}(\mathcal{P}-1)\big)\\
&+\sinh(k)\sinh(\lambda_2)\big(i\omega k^2(\mathcal{M}+8i\omega\mathcal{P}(\mathcal{P}-1))-i\omega^3\mathcal{P}(\mathcal{P}-1)+2k^4(\mathcal{M}+4i\omega\mathcal{P}(\mathcal{P}-1))\big)\Big)\\
&+k^3\lambda_3\mathcal{M}(\mathcal{B}+k^2)(\lambda_2\sinh(k)-k\sinh(\lambda_2))+\sinh(\lambda_3)\Big(\lambda_2k\big(\mathcal{M}k^2(2\mathcal{P}-1)(\mathcal{B}+k^2)-8\omega^2\mathcal{P}^2k^2\mathcal{C}\mathcal{F}(\mathcal{P}-1)\\
&-4i\omega^3\mathcal{P}^2(\mathcal{P}-1)\mathcal{C}\mathcal{F}\big)+\sinh(k)\sinh(\lambda_2)\big(k^4(\mathcal{B}+k^2)(2\mathcal{P}-1)\mathcal{M}+i\omega\mathcal{P}\mathcal{M}k^2(\mathcal{B}+k^2)-8\omega^2\mathcal{P}^2(\mathcal{P}-1)\mathcal{F}\mathcal{C}k^4\\
&-8i\omega^3\mathcal{P}^2(\mathcal{P}-1)\mathcal{F}\mathcal{C}k^2+\omega^2\mathcal{P}^2(\mathcal{P}-1)\mathcal{F}\mathcal{C}\big)\Big)-\lambda_2\cosh(\lambda_2)\Big((\mathcal{B}+k^2)\lambda_3\big(\mathcal{M}k^2-\omega^2\mathcal{P}(\mathcal{P}-1)\big)\cosh(\lambda_3)\sinh(k)\\
&+i\omega\mathcal{P}\big(2\mathcal{C}\mathcal{M}\lambda_3k^3+\sinh(k)\sinh(\lambda_3)(i\omega(\mathcal{P}-1)(\mathcal{B}+k^2)\mathcal{F}+\mathcal{C}\mathcal{M}k^2(2k^2+i\omega\mathcal{P}))\big)\Big)\Big]\\
&+\cosh(k)\Big[\lambda_2\cosh(\lambda_2)\Big(i\omega\mathcal{P}\mathcal{C}\lambda_3\cosh(\lambda_3)\big(i\omega^3\mathcal{P}(\mathcal{P}-1)-2k^4(\mathcal{M}+4i\omega\mathcal{P}(\mathcal{P}-1))\\
&-i\omega k^2(\mathcal{M}+4i\omega\mathcal{P}(\mathcal{P}-1))\big)-\sinh(\lambda_3)\big(k^4\mathcal{M}(\mathcal{B}+k^2)(2\mathcal{P}-1)-8\omega^2\mathcal{P}^2(\mathcal{P}-1)\mathcal{F}\mathcal{C}k^4-4i\omega^3\mathcal{P}(\mathcal{P}-1)\mathcal{F}\mathcal{C}k^2\\
&+\omega^4\mathcal{P}^2(\mathcal{P}-1)\mathcal{F}\mathcal{C}\big)\Big)+k^2\Big((\mathcal{B}+k^2)\lambda_3(\mathcal{M}k^2-\omega^2\mathcal{P}(\mathcal{P}-1))\cosh(\lambda_3)\sinh(\lambda_2)+i\omega\mathcal{P}\big(\mathcal{C}\mathcal{M}\lambda_2\lambda_3(2k^2+i\omega)\\
&+\sinh(\lambda_2)\sinh(\lambda_3)(i\omega(\mathcal{P}-1)\mathcal{F}(\mathcal{B}+k^2)+\mathcal{C}\mathcal{M}(i\omega+k^2)(i\omega\mathcal{P}+2k))\big)\Big)\Big]\Big\}=0
\end{aligned}\tag{A1}$$

Taking the stationary limit of this expression (while neglecting the five trivial roots $\omega = 0$) results in the neutral curve given in Eq. (94).

Using the same procedure for the adjoint problem, yields the implicit dispersion relation in the adjoint space (using $\bar{\lambda}_1 = k$)

$$\begin{aligned}
\bar{\mathcal{D}}(\bar{\omega},\omega,k,\mathcal{M}) \equiv\ & \mathcal{P}(\mathcal{P}-1)i\bar{\omega}^3\Big\{-\Big[\bar{\lambda}_3\cosh(\bar{\lambda}_3)\Big(\bar{\lambda}_2\cosh(\bar{\lambda}_2)\sinh(k)\big(i\bar{\omega}^3k^2(\mathcal{B}+k^2)(\mathcal{P}-1)\mathcal{P}\\
&+i\omega A\mathcal{C}\mathcal{F}(2k+i\bar{\omega}(\mathcal{P}-1))\big)+k\cosh(k)\big(i\bar{\omega}^2\omega\mathcal{C}\mathcal{P}^2(\bar{\omega}^2+4i\bar{\omega}k^2-8k^2)\cosh(\bar{\lambda}_2)-(i\omega A\mathcal{C}\mathcal{F}(2k^2+i\bar{\omega}(\mathcal{P}-1))\\
&+i\bar{\omega}^3k^2(\mathcal{B}+k^2)\mathcal{P}(\mathcal{P}-1))\sinh(\bar{\lambda}_2)\big)+i\omega\bar{\omega}^2\mathcal{C}k^2\mathcal{P}^2(\mathcal{P}-1)\big(8k^3\bar{\lambda}_2-4i\bar{\omega}k\bar{\lambda}_2+(8k^4-8i\bar{\omega}k^2-\bar{\omega}^2)\sinh(k)\\
&\times\sinh(\bar{\lambda}_2)\big)\Big)+k\mathcal{F}\Big(i\omega A\mathcal{C}\mathcal{F}\big((2k^2-i\bar{\omega})\sinh(\bar{\lambda}_2)-2k\bar{\lambda}_2\sinh(k)\big)+\sinh(\bar{\lambda}_3)\big(i\omega\mathcal{C}\bar{\lambda}_2(4k^2(\mathcal{P}-1)\mathcal{P}^2(2k^2-i\bar{\omega})\bar{\omega}^2\\
&+A(k^2(2-4\mathcal{P})+i\bar{\omega}\mathcal{P})+\cosh(k)\cosh(\bar{\lambda}_2)(2Ak^2(2\mathcal{P}-1)-i\bar{\omega}(\mathcal{P}-1)A+\bar{\omega}^2\mathcal{P}^2(\mathcal{P}-1)(\bar{\omega}^2+4i\bar{\omega}k^2-8k^2)))\\
&+i\bar{\omega}^3\bar{\lambda}_2k(\mathcal{B}+k^2)(\mathcal{P}-1)\mathcal{P}-i\bar{\omega}^3k^2\mathcal{P}(\mathcal{P}-1)(\mathcal{B}+k^2)\cosh(k)\sinh(\bar{\lambda}_2)-i\omega k\mathcal{C}\sinh(k)\sinh(\bar{\lambda}_2)(A\\
&\times(i\bar{\omega}(3\mathcal{P}-1)+k^2(4\mathcal{P}-1)+\bar{\omega}^2\mathcal{P}^2(\mathcal{P}-1)(\bar{\omega}^2+8i\bar{\omega}k-8k^4))))\Big)\Big]+\frac{\mathcal{P}\hat{\theta}-1}{\mathcal{P}D\hat{w}}k^2\mathcal{M}\Big[\cosh(\bar{\lambda}_3)\bar{\lambda}_3\\
&\times\Big(\bar{\lambda}_2\big(i\omega A\mathcal{C}\mathcal{F}(\mathcal{P}-1)-2\bar{\omega}^2k^3\mathcal{P}(\mathcal{B}+k^2)+2\bar{\omega}^2\mathcal{P}^2(\mathcal{B}+k^2)\big)+\sinh(k)\cosh(\bar{\lambda}_2)\big(i\omega A\mathcal{C}\mathcal{F}k^2(2\mathcal{P}-1)\\
&+\omega\bar{\omega}A\mathcal{C}\mathcal{F}(\mathcal{P}-1)+2\bar{\omega}^2k^4\mathcal{P}(\mathcal{B}+k^2)-2\bar{\omega}^2k^4\mathcal{P}^2(\mathcal{B}+k^2)+i\bar{\omega}^3k^4(\mathcal{B}+k^2)-i\bar{\omega}^3k^2\mathcal{P}^2(\mathcal{B}+k^2)\big)\Big)\\
&+\bar{\omega}^2k^2(\mathcal{B}+k^2)\mathcal{F}\mathcal{P}(\mathcal{P}-1)\big(2k\bar{\lambda}_2+(2k^2-i\bar{\omega})\sinh(k)\sinh(\bar{\lambda}_2)\big)\sinh(\bar{\lambda}_3)-i\omega\mathcal{C}k\bar{\lambda}_2\cosh(\bar{\lambda}_2)\\
&\times\big(A\mathcal{F}\bar{\lambda}_3+k\sinh(\bar{\lambda}_3)(A\mathcal{F}\sinh(\bar{\lambda}_3)-\bar{\omega}_3\mathcal{P}^2(\mathcal{P}-1)(\bar{\lambda}_3\cosh(\bar{\lambda}_3)+\mathcal{F}\sinh(\bar{\lambda}_3)))\big)-k\cosh(k)\Big(\bar{\lambda}_2\cosh(\bar{\lambda}_2)\\
&\times\big(\bar{\lambda}_3(i\omega A\mathcal{C}\mathcal{F}(2\mathcal{P}-1)+2\bar{\omega}k^2(\mathcal{B}+k^2)(\mathcal{P}-1)\mathcal{P})\cosh(\bar{\lambda}_3)+2\bar{\omega}^2k^2\mathcal{F}\mathcal{P}(\mathcal{P}-1)(\mathcal{B}+k^2)\sinh(\bar{\lambda}_3)\big)\\
&-i\omega\mathcal{C}\big(\sinh(\bar{\lambda}_2)(k^2-i\bar{\omega})(A\mathcal{F}\sinh(\bar{\lambda}_3)+i\bar{\omega}^3\mathcal{P}(\mathcal{P}-1)(\bar{\lambda}_3\cosh(\bar{\lambda}_3)+\mathcal{F}\sinh(\bar{\lambda}_3)))\big)\Big)\Big]\Big\}=0
\end{aligned}\tag{A2}$$

Here $\bar{\mathcal{D}}$ stills contains ω, the frequency of surface waves in the original space. To find the relation between $\bar{\omega}$ and ω is not as simple as in the case of the Rosensweig instability. However, all what we need is to guarantee that $\bar{\omega}$ vanishes at the linear threshold of the physical problem, where $\omega = 0$. The reason for this requirement is that the resonance condition for a nonlinear expansion of the basic equations cannot be satisfied in the case of a finite adjoint frequency $\bar{\omega}$ but a vanishing frequency ω. In the main text the expansion of $\bar{\mathcal{D}}$ in terms of ω is given, Eq.(101), and will not be repeated here. As stated already above, the adjoint dispersion relation depends on the original eigenvectors due to the dynamic bulk coupling between the temperature and the velocity field. The stationary limit for the latter one is given by

$$v_z(z) = \frac{8\mathcal{M}_c k^3 \cosh(k)}{\mathcal{P}N}\Big\{kz\sinh(k)\cosh(kz) - \big[kz\cosh(k) + \sinh(k) - z\sinh(k)\big]\sinh(kz)\Big\}\xi \tag{A3}$$

while the stationary eigenvector for the temperature field reads

$$\begin{aligned}\theta(z) \;=\; & \frac{1}{\mathcal{P}\mathrm{N}}\Big\{2k^2\mathcal{M}\cosh(k)\big(kz\cosh(k) - (z-3)\sinh(k)\big)z\cosh(kz) \\ & - \Big[16k^2\mathcal{F} + \mathcal{M}\big(k^2(1+z) - 1 + (1+k^2(1+z))\cosh(2k)\big) + k\big(\mathcal{M}(1-z+k^2z^2) - 8\mathcal{F}\big)\sinh(2k)\Big]\sinh(kz)\Big\}\xi\end{aligned} \tag{A4}$$

with the abbreviation

$$\mathrm{N} = 2k\big((\mathcal{M}_c - 8)k^2 - 2\mathcal{F}\big)\cosh(k) + 4k\mathcal{F}\cosh(3k) + \big(8(1-2\mathcal{F})k^2 + \mathcal{M}_c + (8k^2 - \mathcal{M}_c)\cosh(2k)\big)\sinh(k) \tag{A5}$$

We can substitute Eqs. (A3) and (A4) into the constant contribution $\bar{\mathcal{D}}_0$ of Eq. (101) resulting in the explicit expression

$$\bar{\mathcal{D}}_0(\bar{\omega}, \omega=0, k, \mathcal{M}) = \bar{\omega}(\omega=0)k^2(B^2+k^2)(\mathcal{P}-1)^2 \quad \mathcal{P}^3\Big(k\cosh(k) + \mathcal{F}\sinh(k)\Big)\frac{2k\sinh^2(k)(\sinh(2k) - 2k)}{1 + 2k^2 - \cosh(2k)} \tag{A6}$$

When assuming $\mathcal{P} \neq 1$ and $k \neq 0$, $\bar{\mathcal{D}}_0 = 0$ can only be satisfied if $\bar{\omega}(\omega = 0) = 0$. Thus, for a stationary instability in the original case, also the adjoint case is stationary.

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