Statistical Physics and Computer Simulations
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Exercise Sheet 1

1. Consider the Hamiltonian of the harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2,$$

where p is the particle momentum, x the coordinate, m the mass and ω the frequency.

   (a) The phase space trajectory is an ellipse. What is the area enclosed by the ellipse (the so–called "phase space volume")?

   (b) Let

$$P = \sigma p, \qquad X = \sigma^{-1} x,$$

   where σ is some constant factor. Show that this transformation is canonical and that it conserves the phase space volume.

   (c) Choose σ such that the new trajectory is just a circle in phase space.

   (d) Let

$$P = A I^{1/2} \cos\varphi, \qquad X = A I^{1/2} \sin\varphi.$$

   φ is the phase space angle ("angle variable"), and I is the corresponding so–called "action variable". We interpret φ as a new coordinate, and I as a new momentum, while A is a constant prefactor. Assume for the time being that the transformation is canonical. Under this assumption, derive the equations of motion for φ and I, and solve them. Establish a relation the total energy of the oscillator. Show that A can be chosen appropriately such that the transformation is indeed canonical.

   (e) Draw the trajectory in (φ, I)–space and calculate again the enclosed phase space volume.

2. Consider a rope of length L with mass per unit length μ, suspended between two points (x_0, y_0) and (x_1, y_1). Gravity (acceleration g) points in $-y$ direction. At rest, the potential energy must be minimum. Show

via variational calculus that the rope has a cosh shape. *Hints:* (i)There is one consraint, given by the fixed length of the rope. Make yourself clear that this involves a Lagrange multiplier in the sense of ordinary calculus, not of variational calculus. (ii) Use the energy theorem of Lagrangian mechanics.

3. Consider a mathematical pendulum with one degree of freedom φ and Lagrangian

$$L = \frac{1}{2} m l^2 \dot{\varphi}^2 + mgl \cos\varphi .$$

   (a) Transform to Hamilton formalism, and sketch the trajectories in phase space.

   (b) Discuss the special case where the energy is just sufficient to lift the pendulum up to the top vertex $\varphi = \pi$. Show that in that case the pendulum needs an infinite amount of time to get there. What special point does this correspond to in the phase space plot?