


Macroscopic behavior of the distorted A and B phases and the polar phase of superfluid ^3He in an anisotropic aerogel

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We describe the macroscopic behavior of superfluid ^3He in the polar and the distorted A and B phases in an anisotropic aerogel. It turns out that the distorted A phase shares many features with superfluid ^3He - A . Its order parameter contains the polar phase as well as the A phase as a smooth limiting case. The former is in agreement with the fact that the phase transition polar to distorted A is observed to be of second order. Due to the different structure of the order parameter, the unit vector \hat{l} , which is characteristic of the superfluid ^3He - A and the distorted A phase, is not present in the polar phase. As new macroscopic variables compared to superfluid ^3He we have for the superfluid distorted A phase in an aerogel the strain field associated with the aerogel network. In addition, we have in superfluid ^3He relative rotations between the anisotropic gel network and the \hat{l} vector as a macroscopic variable. As a result of these additional macroscopic variables we find new static and dynamic cross-coupling terms, which come as reversible (zero entropy production) as well as as irreversible contributions. The polar phase in an anisotropic aerogel is anisotropic in its elastic properties. We have as additional macroscopic variables relative rotations similar to those of a nematic liquid crystalline elastomer, since the preferred direction in the polar phase is even under time reversal and not odd as for the distorted A phase. These additional features lead to static and dissipative cross-coupling terms, which cannot exist for a polar phase in an isotropic aerogel. For the distorted B phase we find for the orbital part a macroscopic dynamic behavior which shares some features with that of the polar phase. In contrast to the polar phase, however, the distorted B phase does not show relative rotations, since there is only one preferred direction. The preferred direction in the distorted B phase leads to a preferred direction in spin space as well. As a consequence the spin waves become anisotropic and longitudinal and transverse spin waves become coupled, even in the absence of a magnetic field. We also discuss briefly the coupling to the spin degrees of freedom in the polar and the distorted A phase.

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I. INTRODUCTION

A few years ago the long sought polar phase of superfluid ^3He has been found using NMR [1] in strongly anisotropic aerogels called nafen [2–4]. In the following this phase has also been identified using torsional oscillator measurements [5] and its properties have been further elucidated experimentally in detail using CW and pulsed NMR techniques [6–8]. Quite recently the polar phase has also been identified in a second class of strongly anisotropic aerogels called mullite [9].

Prior to the discovery of the polar phase, polar distorted A and B phases of superfluid ^3He had been identified in so-called Obninsk aerogels [10–12]. It was found that nafen-90 [1,6] can show up to three superfluid phases as a function of temperature and pressure: polar distorted B , polar distorted A ,

and the genuine polar phase. In the following we will use throughout the notation distorted B and distorted A and leave the notation polar to the original notation of the polar phase [13–15].

In parallel to the various experimental developments the modeling of the various superfluid phases in an anisotropic aerogel has been advanced [16–18]. In addition, it had been shown previously that the polar phase could be stable in anisotropic aerogels [19] and channels [20]. Quite recently an experimental/theoretical cooperation has elucidated in detail [7,21] the important role played by half-quantum vortices and walls in the polar and the distorted phases of superfluid ^3He .

The question of disorder-induced states in superfluid ^3He such as the LIM state is an important one for aerogels [22]. In an aerogel that is globally isotropic it has been shown, using NMR, that long range orientational order can be destroyed giving rise to an orientational glass in the superfluid A phase [23,24]. In [24] it has also been pointed out that ^3He - A generated by warming from superfluid ^3He - B can have long range orientational order.

We note that the field of superfluid ^3He in aerogels also has a prehistory using isotropic aerogels. Experiments on superfluid ^3He in isotropic aerogels were performed using torsional oscillator techniques [25,26] and pulsed nuclear magnetic resonance (NMR) [27–30]. The torsional oscillator experiments

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showed that the transition temperature and superfluid fraction are strongly suppressed from the bulk value [25]. In spite of the randomness of the strand spacing inside the aerogel, a sharp phase transition can be observed which leads to the assumption that a homogeneous phase exists. Early theoretical investigations of superfluid ^3He in an aerogel have been described by Volovik [31] and by Yip, Thuneberg, and Sauls [32]. The NMR results provide evidence that the existing phase in the aerogel is a phase with equal spin pairing (ESP), but from the tipping angle dependency it had been deduced [27–30] that it might not be the A phase. Since the polar phase [13–15] emerged as a natural alternative option, this stimulated work on the hydrodynamics of the polar phase in an isotropic aerogel [33,34]. Among other features it was found that the hydrodynamics of the spin part is isomorphic to that of ^3He - A [35] and of ^3He - A_1 [36–38]. Assuming an oriented spatially homogeneous phase, the orbital part of the hydrodynamics turned out to be that of a superfluid uniaxial nematic liquid crystal.

In this paper we discuss the influence of an anisotropic aerogel on the hydrodynamics of the polar phase and the distorted A and B phases. We model the aerogel as an elastic continuum that also provides a preferred direction. The static and dynamic couplings of those quantities to the standard variables of the three phases under consideration are the main topic of the paper. It is organized as follows. In Sec. II A we present the relevant variables due to conservation laws, in Sec. II B due to broken symmetries, in particular Sec. II B 1 those resulting from the structures of the polar phase, in Sec. II B 2 the distorted A phase, and in Sec. II B 3 the distorted B phase, and finally in Sec. II C the slowly relaxing rotations of the appropriate structures relative to the elastic network. The macroscopic orbital dynamics of the polar phase in the presence of an elastic aerogel and of the distorted A and B phase are discussed in detail in Secs. III, IV, and V A, respectively. For the latter case the spin dynamics is dealt with in Sec. V B. Comments on the effects of the presence of the aerogel on the spin dynamics of the polar and the distorted A phase are presented at the end of the appropriate Secs. III and IV. In Sec. VI we give a summary. In the Appendix we analyze the influence of the presence of a *transient* network on the orbital dynamics of the polar phase and the distorted A and B phases.

In this paper we aim at a most general description possible, e.g., we allow for relative rotations among the preferred direction of the aerogel and the orbital direction in the polar phase to be a slowly relaxing macroscopic variable, but we discuss at the end of Sec. III D how our description simplifies when those relative rotations are irrelevant on the macroscopic scale. Similarly, we assume the distorted A phase to be biaxial in orbital space, but discuss in Sec. IV E how a uniaxial situation, e.g., in disorder-induced LIM states [22], is contained in our description.

II. THE RELEVANT MACROSCOPIC VARIABLES FOR THE DISTORTED A , B , AND THE POLAR PHASE IN THE PRESENCE OF AN ELASTIC AEROGEL

In this paper we will use linearized hydrodynamics [39] to describe the macroscopic behavior of superfluid ^3He in

the polar phase as well as in the distorted A and B phases in the presence of an elastic network. We will derive the balance equations describing the behavior of the system in the low frequency, long wavelength limit. Low frequencies in this context means small compared to all collisional frequencies, while wavelengths are considered to be long if they are large compared to all microscopic lengths. Naturally these conditions for the purely hydrodynamic regime impose rather severe constraints on the frequencies and wave vectors for which this approach is strictly valid. Nevertheless, the hydrodynamic description and its generalization to include variables that relax on a long, but finite timescale have turned out to be rather useful [40].

A. Hydrodynamic variables due to conservation laws

The conserved quantities in superfluid ^3He are ρ (mass density), ε (energy density), and g_i (momentum density) just as in any normal fluid. The Latin indices refer to vector components in a suitable Cartesian frame (orbital space). All variables are related to the entropy density σ by the Gibbs relation

$$d\varepsilon_c = T d\sigma + \mu d\rho + v_i^n dg_i, \quad (1)$$

thereby defining the thermodynamic quantities, temperature T , chemical potential μ , and normal velocity v_i^n .

B. Hydrodynamic variables due to internal structures that break symmetries

We will derive the hydrodynamics of various superfluid ^3He phases in the presence of an anisotropic aerogel. We model the macroscopic influence of such an aerogel by an elastic strain variable ϵ_{ij} . The aerogel breaks translational symmetry, which gives rise to a translation vector as symmetry variable. To exclude homogeneous translations and homogeneous rotations, the strain is, in linear order, given by $\epsilon_{ij} = (1/2)(\nabla_i u_j + \nabla_j u_i)$. Elastic strains enter the energy density by

$$d\varepsilon_e = \Psi_{ij} d\epsilon_{ij}, \quad (2)$$

thereby defining the elastic stresses Ψ_{ij} .

In this connection we note that in Ref. [41] the coupling between the superfluid order parameter and the strain field has been studied close to phase transition temperatures in the framework of linearized Ginzburg-Landau equations.

The anisotropy of the elastic network is described by a unit vector ζ_i . Its role as a variable will be discussed in Sec. II C.

Since the ^3He atoms have spin $\frac{1}{2}$ each, there is the magnetization density s_v . The frame to describe the orientation of spins is *a priori* not the same as that of, e.g., the flow variables. Therefore it is customary to use in “spin space” a different Cartesian frame indicated by Greek indices. Neglecting the tiny magnetic dipole-dipole interaction (spin-orbit coupling), the orientations of spin and orbital space are independent and the hydrodynamics in orbital and spin space can be developed separately.

In the absence of a magnetic field s_v is a conserved quantity, but acquires a source term in its dynamic equation due to

the field. The energy density

$$d\varepsilon_s = \chi_{\nu\mu} d\nabla_\mu s_\nu + h'_\nu ds_\nu \equiv h_\nu ds_\nu, \quad (3)$$

where h'_ν is zero in the absence of a magnetic field. In linear order one can simply condense the notation by using the conjugate $h_\nu \equiv h'_\nu - \nabla_\mu \chi_{\nu\mu}$.

In superfluid ^3He the neutral He atoms combine to form Cooper pairs similar to those found in superconductors which can be viewed as composite bosons. While the electrons in superconductors are in a spin-singlet s -wave state, the He atoms are in a spin-triplet p -wave state. This fact clearly distinguishes the two situations: The pair of electrons has no internal structure, but the pair of He atoms is intrinsically anisotropic. Because of the spin-triplet and p -wave pairing, the order parameter $T_{\nu j}$ has to be a complex 3×3 matrix whose expectation value can in general be written as [42,43]

$$\begin{aligned} \langle T_{\nu j}(\mathbf{c}, \mathbf{r}) \rangle &= \mathcal{F}(|\mathbf{c}|) A_{\nu j}(\mathbf{r}) e^{i\varphi(\mathbf{r})}, \\ A_{\nu j} A_{\nu j}^* &= 1. \end{aligned} \quad (4)$$

where ν is an index in spin space, j an index in orbital space, \mathbf{r} is the position vector of the center of gravity, and \mathbf{c} is the relative vector between the two He atoms.

The normalization amplitude \mathcal{F} describes the degree of ordering and is considered a microscopic variable, which does not appear in the macroscopic dynamics of the system.

The overall phase factor is characteristic for superfluids and superconductors, because it reflects the fact that gauge invariance is spontaneously broken and the phase variation $\delta\varphi$ has to be used in the hydrodynamic description. Therefore we will use the superfluid velocity

$$v_i^s = \frac{\hbar}{2m} \nabla_i \varphi \quad (5)$$

according to the two-fluid model developed by Landau [39] and Khalatnikov [44] to describe the macroscopic behavior of the system. The conjugate quantity to the superfluid velocity is $\sim \lambda_i^s$ with

$$d\varepsilon_\lambda = \lambda_i^s d v_i^s = (\hbar/2m) \lambda_i^s d \nabla_i \varphi. \quad (6)$$

The macroscopic properties of the different ^3He phases are represented by the matrix $A_{\nu j}$. The only restriction imposed on the complex 3×3 matrix is that of normalization which means that many different structures are theoretically possible and describe different phases of superfluid ^3He [14].

A Landau expansion up to terms of fourth order in $A_{\nu j}$ can be used to determine the relative stability of the respective phases, which shows that in weak-coupling theory only the so-called B phase can be stable. If strong-coupling effects are taken into account, it can be shown that the A phase may also represent the equilibrium state under certain conditions, depending on the values of the parameters in the Landau expansion. Apart from A and B phase, which can be found in zero magnetic field, in high magnetic fields another phase is found, which is called A_1 phase. All three phases have been observed experimentally in the bulk [40].

For slightly different values of the parameters in the Landau expansion, two phases similar to the A phase represent the state with the lowest free energy: the polar phase and the distorted A phase. In addition, one has the distorted B phase.

1. The structure of the polar phase

In the polar phase the matrix $A_{\nu j}$ defined in Eq. (4) takes the form

$$A_{\nu j}(\mathbf{r}) = d_\nu(\mathbf{r}) m_j(\mathbf{r}), \quad (7)$$

where the unit vectors \hat{m} and d_ν are the preferred direction in orbital and spin space, respectively.

For the polar phase the energy gap is zero in the plane normal to m_j and a line node exists [40,45].

As they follow from Eq. (7) via the contractions $A_{\nu i} A_{\nu j} \sim m_i m_j$ and $A_{\nu i} A_{\mu i} \sim d_\nu d_\mu$, a form similar to the nematic order parameter, it is obvious that both, m_i and d_ν are not really vectors but directors, meaning a substitution of m_i with $-m_i$, and of d_ν with $-d_\nu$ must not change the hydrodynamics. They can be viewed as directions that cannot distinguish head from tail. Breaking spontaneously rotational symmetry in spin and orbital space, rotations of them give rise to two hydrodynamic variables in orbital and spin space each that enter the energy density

$$d\varepsilon_p = \Phi_{ij}^m d \nabla_j m_i + \Phi_{\nu\mu}^d d \nabla_\mu d_\nu, \quad (8)$$

defining the conjugate quantities Φ_{ij}^m and $\Phi_{\nu\mu}^d$.

Both preferred directions are even under space inversion, as well as under time reversal. This is in contrast to the A phase, where the preferred direction in orbital space is odd under time reversal. Since m_i and d_ν are unit vectors, there is $m_i \nabla_j m_i = 0 = d_\nu \nabla_\mu d_\nu$.

The small dipole interaction leads to a coupling of the preferred directions in orbital and spin space $d_\nu \perp m_i$. In the following we will disregard this coupling and comment on its effect briefly at the end of Sec. III.

2. The structure of the distorted A phase

In the distorted A phase (sometimes also called DA and pDA phase [1,6,21]) the matrix $A_{\nu j}$ takes the form [1,6]

$$A_{\nu j}(\mathbf{r}) = d_\nu(\mathbf{r}) [a m_j(\mathbf{r}) + i b n_j(\mathbf{r})], \quad (9)$$

where the unit vectors m_i and n_j are mutually orthogonal unit vectors in orbital space and $a^2 + b^2 = 1$. It contains two important special cases: for $a = 1$ (and therefore $b = 0$) the polar phase results, while for $a = 1/\sqrt{2}$ (and therefore $b = a$) one obtains the A phase. Thus, the polar phase and the distorted A phase are connected by a second order phase transition as has been detected experimentally [1,6]. For $1/2 < a^2 < 1$ one gets the distorted A phase.

We note that in the distorted A phase the energy gap is zero along l_i and thus, concerning the energy gap, a smooth (continuous) special case of the A phase [40,45]. Both have a point node in the gap.

From Eq. (9) one can construct a preferred direction in orbital space $\epsilon_{ijk} A_{\nu j} A_{\nu k}^* \sim l_i$, which when normalized

$$l_i = (\mathbf{m} \times \mathbf{n})_i \quad (10)$$

is the same as in the A phase. In particular, it is an axial vector, even under spatial inversion, but is odd under time reversal. The latter property becomes manifest when taking the complex conjugate of $\epsilon_{ijk} A_{\nu j} A_{\nu k}^*$.

The preferred direction in spin space d_v follows from $A_{vi}A_{\mu i}^* \sim d_v d_\mu$ and is a directorlike quantity, similar to the case of the polar phase.

Breaking spontaneously rotational symmetry in spin and orbital space, rotations of d_v and l_i give rise to two hydrodynamic variables in orbital and spin space each that enter the energy density

$$d\varepsilon_A = \Phi_{ij}^l d\nabla_j l_i + \Phi_{v\mu}^d d\nabla_\mu d_v, \quad (11)$$

defining the conjugate quantities Φ_{ij}^l and $\Phi_{v\mu}^d$. The small dipole interaction leads to a coupling of the preferred directions in orbital and spin space $d_v \parallel l_i$. In the following we will disregard this coupling and comment on its effect at the end of Sec. IV. In addition, we will also comment on the changes of the macroscopic dynamics due to the presence of the uniaxial aerogel.

3. The structure of the distorted B phase

The distorted B phase is fully gapped as the isotropic B phase, but the gap has now uniaxial symmetry, which coincides with the preferred direction ζ_i of the uniaxial aerogel.

In the distorted B phase the matrix A_{vj} takes the form [12,17,21]

$$A_{vj} = d_v m_j + q_1 e_v^1 n_j + q_2 e_v^2 k_j, \quad (12)$$

with $\text{mod } q_1, |q_2| \in (0, 1)$ and $|q_1| = |q_2| \equiv q$ describing the relative gap in the plane perpendicular to the nafen strands, which define the preferred direction ζ_i^0 . For $q = 0$ one obtains the order parameter of the polar phase, while $q = 1$ recovers the order parameter of the isotropic B phase.

The vectors \mathbf{m} , \mathbf{n} , and \mathbf{k} form an orthogonal triad of unit vectors in orbital space, and $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{d}}$ in spin space. It is not possible to identify any preferred direction from the two triads, neither in spin nor in orbital space (as in the B phase). Neglecting the very small dipole-dipole interaction, the relative orientation of the two triads in equilibrium, described by a rotation matrix between the two frames n_{vj}^0 , is not fixed energetically. Therefore, deviations δn_{vj} are the hydrodynamic variables (as in the B phase) and enter the energy density as

$$d\varepsilon_B = \Psi_{vjk}^n d\nabla_j n_{vk}. \quad (13)$$

It should be noted that the form of A_{vi} in Eq. (12) is not the most general one, since m_i has been chosen to be along ζ_i . The most general one contains an additional rotation of ζ_i (cf. Eq. (8) in Ref. [12]). However, this is irrelevant for the hydrodynamics, since $\nabla_j n_{vk}$ does not couple to any orbital variable and its absolute orientation in orbit space is irrelevant. Likewise, a rotation of ζ_i (relative to the orbital triad) is not a hydrodynamic variable, but makes the orbit space hydrodynamics uniaxial. The effects of the spin-orbit coupling are dealt with in Sec. VB.

Due to the special properties of a rotation matrix, e.g., $n_{vi}n_{\mu i} = \delta_{v\mu}$ and $n_{vi}n_{vj} = \delta_{ij}$, the δn_{vj} contains three independent variables according to the three spontaneously broken rotational symmetries. They can also be parametrized by a unit vector describing the rotation direction plus the rotation angle.

At the end of Sec. V we will summarize the differences between the macroscopic dynamics of the usual B phase and the distorted B phase in an anisotropic aerogel.

C. Slowly relaxing rotations relative to the elastic network

As additional input from experimental results [1,6] it is known that in the polar phase the preferred direction m_i is on average parallel to the strand direction of the aerogel, which is the preferred direction ζ_i . This condition is not due to a broken symmetry, but results from molecular interaction forces. Therefore, deviations from $m_i^0 \parallel \zeta_i^0$, e.g., $\delta m_i - \delta \zeta_i$, are relaxing variables, which however interact with the hydrodynamic variables on timescales shorter than the relaxation times. We will take into account those slowly relaxing relative rotations as has been done, e.g., for nematic liquid crystal elastomers by de Gennes [46].

This variable can be written in the polar phase as

$$\Omega_i = \delta(\mathbf{m} \times \boldsymbol{\zeta})_i = \epsilon_{ijk} m_j^0 (\delta \zeta_k - \delta m_k) \quad (14)$$

because of $m_i^0 \parallel \zeta_i^0$. Another possible definition is

$$\Omega_i = \delta m_i - \frac{1}{2} \zeta_j^0 (\nabla_i u_j - \nabla_j u_i) \quad (15)$$

involving rotations expressed by antisymmetric gradients of the translation vector.

In the distorted A phase the strand direction ζ_i is parallel to m_i as in the polar phase. However, in the distorted A phase the preferred direction in orbital space is l_i , which is perpendicular to m_i . Thus, in equilibrium $l_i^0 \zeta_i^0 = 0$ and the distorted A phase is biaxial in orbital space. A deviation from the orthogonality

$$\Omega = \zeta_i^0 \delta l_i + l_i^0 \delta \zeta_i \quad (16)$$

is the variable describing relative rotations between l_i and ζ_i . This is a scalar variable that changes sign under time reversal.

The variables Ω_i and Ω represent the dynamics of local fluctuations of the preferred direction of the strands and are independent of the elastic deformations that only contain symmetric gradients of u_i .

The relative rotations lead in the energy density to

$$d\varepsilon_r = W_i d\Omega_i + W d\Omega \quad (17)$$

defining the conjugates W_i and W for the polar and the distorted A phase, respectively.

In the distorted B phase, ζ_i is the preferred direction making this phase uniaxial. Since ζ_i is the only preferred direction, there are no relative rotations possible.

III. THE ORBITAL DYNAMICS OF THE POLAR PHASE IN THE PRESENCE OF AN ANISOTROPIC ELASTIC AEROGEL

In the preceding section we have characterized the hydrodynamic variables characteristic of the polar phase in an anisotropic aerogel, the distorted A phase in an anisotropic aerogel and the distorted B phase in an anisotropic aerogel. Here and the following Secs. IV and V we will derive macroscopic equations for these three superfluid phases making use of linear irreversible thermodynamics including the local formulation of the first and second law of thermodynamics as well as symmetry properties. The latter includes the behavior

under parity, time reversal, Galilei transformations, as well as under rotations and translations.

To derive the full set of dynamic equation is then a two step procedure [47,48]. First one writes down the Gibbs relation, the local formulation of the first law of thermodynamics, for the hydrodynamic variables. This way one defines the thermodynamic conjugate quantities or thermodynamic forces. The thermostatic behavior is then obtained by expanding the generalized energy into the hydrodynamic variables taking into account all symmetry properties. By then taking the (variational) derivative of the generalized energy with respect to the variables one obtains the thermodynamic forces.

In the second step one writes down first the dynamic equations for the three types of variables in macroscopic dynamics: conservation laws for the conserved variables and balance equations for the variables associated with spontaneously broken continuous symmetries and with macroscopic variables, which relax on a long, but finite timescale. These dynamic equations contain currents and quasicurrents associated with the dynamics of the variables. To close the system of equations in the framework of linear irreversible thermodynamics one then expresses the current and quasicurrents in relations linear in the thermodynamic forces. In addition, one splits all currents and quasicurrents into reversible (no entropy generation) and into irreversible contributions [positive entropy (or heat) generation]. The irreversible contributions in the currents and quasicurrents can be derived from a dissipation function, which is an expansion quadratic in the thermodynamic forces, by taking a (variational) derivative with respect to the thermodynamic forces. When splitting the currents and quasicurrents into reversible and irreversible contributions the behavior under time reversal plays a crucial role.

In this section on the derivation of the macroscopic dynamics of the polar phase in an anisotropic aerogel we will comment in some detail on the points outlined above. In the next two sections, Secs. IV and V, we will carry out the same program for the distorted *A* phase and for the distorted *B* phase. It seems worthwhile to mention that this program has been carried out before for the three superfluid phases arising in ^3He in the bulk: the superfluid *A* phase [35,43,49–51] and the superfluid *B* phase [43,52] as well as for the superfluid *A*₁ phase [36–38], which arises only in a magnetic field.

Throughout the bulk of this section we focus on the orbital dynamics of the polar phase in the presence of an anisotropic elastic aerogel and comment on the influence of the spin degrees of freedom and their coupling to the orbital degrees of freedom in Sec. III E.

A. Statics and thermodynamics

To obtain the static properties of our system we formulate the local first law of thermodynamics relating changes in the entropy density σ to changes in the hydrodynamic and macroscopic variables discussed above. According to the discussions in Sec. II, Eqs. (1), (2), (6), (8), and (17), we get the Gibbs relation for the variables acting in orbital space

$$d\varepsilon = Td\sigma + \mu d\rho + v_i^n dg_i + \lambda_i^s dv_i^s + \Phi_{ij}^m d(\nabla_j m_i) + \Psi_{ij} d\epsilon_{ij} + W_i d\Omega_i, \quad (18)$$

where the thermodynamic conjugates are defined as partial derivatives of the energy density with respect to the appropriate variables [48]. Let us list the symmetry properties used: Scalar quantities are ε , σ , and ρ , while g_i and v_i^s are polar vectors, Ω_i is an axial vector, and ϵ_{ij} and $\nabla_j m_i$ are tensors of second rank. Even under time reversal are ε , σ , Ω_i , ϵ_{ij} , and $\nabla_j m_i$, while g_i and v_i^s are odd under time reversal. Odd under parity are v_i^s , g_i , and $\nabla_j m_i$, while ε , σ , ρ , ϵ_{ij} , and Ω_i are even under parity. The behavior of the thermodynamic conjugates defined via Eq. (18) under time reversal and parity can then be read off immediately.

In writing down Eq. (18) we have discarded the part of Eq. (8) associated with the spin degrees of freedom, since the focus is on the orbital degrees of freedom. Compare, however, also the discussion in Sec. III E.

To determine the thermodynamic conjugate variables we need an expression for the local energy density. This energy density must be invariant under time reversal as well as under parity and it must be invariant under rigid rotations, rigid translations and covariant under Galilei transformations. In addition to that this energy density must have a minimum, because there exists an equilibrium state for the gel. Therefore the expression for the energy density needs to be convex. Taking into account these symmetry arguments we write down an expansion for the generalized energy density up to second order in the variables that describe deviations out of that equilibrium:

$$\begin{aligned} \varepsilon = & \frac{1}{2} \rho_0 \left(\frac{\rho^s}{\rho^n} \right)_{ij} v_i^s v_j^s + \frac{1}{2} \left(\frac{1}{\rho^n} \right)_{ij} g_i g_j \\ & - \left(\frac{\rho^s}{\rho^n} \right)_{ij} v_i^s g_j + \frac{1}{2} \mu_{ijkl} \epsilon_{ij} \epsilon_{kl} \\ & + \frac{1}{2} K_{ijkl} (\nabla_j m_i) (\nabla_l m_k) + \frac{1}{2} D_1 \Omega_i \Omega_i \\ & + D_2 (m_j^0 \delta_{ik}^\perp + m_k^0 \delta_{ij}^\perp) \Omega_i \epsilon_{jk} + c_{\rho\sigma} (\delta\rho) (\delta\sigma) \\ & + \sigma_{ijk}^\sigma (\nabla_j m_i) (\nabla_k \sigma) + \sigma_{ijk}^\rho (\nabla_j m_i) (\nabla_k \rho) \\ & + \epsilon_{ij} (\chi_{ij}^\sigma \delta\sigma + \chi_{ij}^\rho \delta\rho) + c_{\rho\rho} (\delta\rho)^2 + c_{\sigma\sigma} (\delta\sigma)^2, \quad (19) \end{aligned}$$

with $\delta_{ij}^\perp = \delta_{ij} - m_i^0 m_j^0$. A δ denotes deviations from the (constant) equilibrium value of the appropriate variable, e.g., $\delta\rho \equiv \rho - \rho_0$.

Apart from the energy density of a normal fluid, Eq. (19) contains anisotropic superfluid kinetic energies, the gradient energy of the \hat{m} vector as well as the elastic energy and the relative rotations between m_i and ζ_i , the anisotropy axis of the aerogel.

Because we discuss a uniaxial system, the tensors have more independent constants than in the isotropic case. The two-fluid density tensors are

$$\left(\frac{\rho^s}{\rho^n} \right)_{ij} = \frac{\rho_\perp^s}{\rho_\perp^n} \delta_{ij}^\perp + \frac{\rho_\parallel^s}{\rho_\parallel^n} m_i^0 m_j^0, \quad (20)$$

$$\left(\frac{1}{\rho^n} \right)_{ij} = \frac{1}{\rho_\perp^n} \delta_{ij}^\perp + \frac{1}{\rho_\parallel^n} m_i^0 m_j^0. \quad (21)$$

The tensor of the elastic energy for example now has five independent constants instead of only two and takes the form

$$\begin{aligned} \mu_{ijkl} = & \mu_1 \delta_{ij}^\perp \delta_{kl}^\perp + \mu_2 (\delta_{ik}^\perp \delta_{jl}^\perp - \frac{1}{2} \delta_{ij}^\perp \delta_{kl}^\perp + \delta_{il}^\perp \delta_{jk}^\perp - \frac{1}{2} \delta_{ij}^\perp \delta_{kl}^\perp) \\ & + \mu_3 m_i^0 m_j^0 m_k^0 m_l^0 + \mu_4 (m_i^0 m_j^0 \delta_{kl}^\perp + m_k m_l \delta_{ij}^\perp) \\ & + \mu_5 (m_i^0 m_k^0 \delta_{jl}^\perp + m_i^0 m_l^0 \delta_{jk}^\perp + m_j^0 m_k^0 \delta_{il}^\perp + m_j^0 m_l^0 \delta_{ik}^\perp), \end{aligned} \quad (22)$$

while the tensor K_{ijkl} is of the same symmetry as the tensor for the Frank elastic coefficients in nematic liquid crystals

$$K_{ijkl} = K_1 \delta_{ij}^\perp \delta_{kl}^\perp + K_2 m_p^0 \epsilon_{pij} m_q^0 \epsilon_{qkl} + K_3 m_j^0 m_l^0 \delta_{ik}^\perp \quad (23)$$

and has, thus, three independent constants describing splay, twist, and bend deformations of the \mathbf{m} vector.

There are two contributions to the energy density concerning relative rotations. One is proportional to D_1 and the other proportional to D_2 . One can interpret these coefficients as a measure for the coupling strength of the aerogel network to the \mathbf{m} vector, although the microscopic mechanism of this interaction is not precisely understood so far.

Finally, we are left with the couplings between the scalars ρ and σ and the strain field as well as with the coupling between the gradient of the scalars and the gradient of the \mathbf{m} vector. The tensors take the following form, respectively:

$$\chi_{ij}^\xi = \chi_{||}^\xi m_i^0 m_j^0 + \chi_{\perp}^\xi \delta_{ij}^\perp, \quad (24)$$

$$\sigma_{ijk}^\xi = \sigma^\xi (\delta_{ik}^\perp m_j^0 + \delta_{ij}^\perp m_k^0), \quad (25)$$

where ξ can be either ρ or σ .

We also give the expressions for the conjugated variables in terms of the hydrodynamic and macroscopic variables. They are defined as the partial derivative of the energy density with respect to the appropriate variable, while all the other variables are kept constant, denoted by dots at the brackets in the following. We obtain (omitting second order gradients)

$$g_i = \rho_{ij}^n v_j^n + \rho_{ij}^s v_j^s, \quad (26)$$

$$\lambda_i^s = \left(\frac{\partial \varepsilon}{\partial v_i^s} \right)_{\dots} = g_i - \rho_0 v_i^n, \quad (27)$$

$$\Phi_{ij}^m = \left(\frac{\partial \varepsilon}{\partial (\nabla_j m_i)} \right)_{\dots} = K_{ijkl} \nabla_l m_k + \sigma_{ijk}^p \nabla_k \rho + \sigma_{ijk}^s \nabla_k \sigma, \quad (28)$$

$$\begin{aligned} \Psi_{ij} = \left(\frac{\partial \varepsilon}{\partial \epsilon_{ij}} \right)_{\dots} = & + \mu_{ijkl} \epsilon_{kl} + \chi_{ij}^\sigma \delta \sigma + \chi_{ij}^\rho \delta \rho \\ & + D_2 (m_j^0 \delta_{ik}^\perp + m_i^0 \delta_{kj}^\perp) \Omega_k, \end{aligned} \quad (29)$$

$$W_i = \left(\frac{\partial \varepsilon}{\partial \Omega_i} \right)_{\dots} = D_1 \Omega_i + D_2 (m_j \delta_{ik}^\perp + m_k \delta_{ij}^\perp) \epsilon_{jk}, \quad (30)$$

$$\delta T = \left(\frac{\partial \varepsilon}{\partial \delta \sigma} \right)_{\dots} = \chi_{ij}^\sigma \epsilon_{ij} + 2c_{\sigma\sigma} \delta \sigma + c_{\rho\sigma} \delta \rho - \sigma_{ijk}^\sigma \nabla_j \nabla_k m_i, \quad (31)$$

$$\delta \mu = \left(\frac{\partial \varepsilon}{\partial \delta \rho} \right)_{\dots} = \chi_{ij}^\rho \epsilon_{ij} + 2c_{\rho\rho} \delta \rho + c_{\rho\sigma} \delta \sigma - \sigma_{ijk}^\rho \nabla_j \nabla_k m_i, \quad (32)$$

where Eq. (26) follows from $v_i^n \equiv \partial \varepsilon / \partial g_i$. We also note that Galilean invariance requires $\rho_0 \delta_{ij} = \rho_{ij}^n + \rho_{ij}^s$, where ρ_0 is the

equilibrium value of the density ρ . This relation has been used to simplify Eq. (27).

B. Dynamic equations

To determine the dynamics of the variables we take into account that the first class of our set of variables, the conserved quantities, obey a local conservation law, while the dynamics of the other two classes of variables can be described by a simple balance equation, where the counterterm to the temporal change of the quantity is called a quasicurrent. For the set of dynamical equations we get

$$\dot{\rho} + \nabla_i g_i = 0, \quad (33)$$

$$\dot{\sigma} + \nabla_i j_i^\sigma = \frac{2R}{T}, \quad (34)$$

$$\dot{g}_i + \nabla_j (\delta_{ij} p + \sigma_{ij}) = 0, \quad (35)$$

$$\dot{m}_i + \epsilon_{ijk} m_j^0 \omega_k + X_i^m = 0, \quad (36)$$

$$(\hbar/2m)\dot{\varphi} + I_\varphi = 0, \quad (37)$$

$$\dot{\epsilon}_{ij} + Y_{ij} = 0, \quad (38)$$

$$\dot{\Omega}_i + Z_i = 0, \quad (39)$$

with $\omega_i = (1/2)\epsilon_{ijk}\nabla_j v_k^n$ the vorticity. The entropy production R/T , with R the dissipation function, acts as a source term in Eq. (34). The pressure p in Eq. (35) is given by $-\partial E / \partial V$, with E the total energy, cf. Ref. [48], and reads for our system

$$p = -\varepsilon + \mu \rho + T \sigma + v_i^n g_i. \quad (40)$$

The time derivatives in Eqs. (33)–(39) have the following behavior under time reversal: $\dot{\rho}$, $\dot{\sigma}$, \dot{m}_i , $\dot{\epsilon}_{ij}$, and $\dot{\Omega}_i$ are odd, while \dot{g}_i and $\dot{\varphi}$ are even under time reversal.

Now we can decompose all currents and quasicurrents listed in Eqs. (33)–(39) into reversible and irreversible contributions:

$$j_i^\sigma = j_i^{\sigma R} + j_i^{\sigma D}, \quad (41)$$

$$\sigma_{ij} = \sigma_{ij}^R + \sigma_{ij}^D, \quad (42)$$

$$X_i^m = X_i^{mR} + X_i^{mD}, \quad (43)$$

$$I_\varphi = I_\varphi^R + I_\varphi^D, \quad (44)$$

$$Y_{ij} = Y_{ij}^R + Y_{ij}^D, \quad (45)$$

$$Z_i = Z_i^R + Z_i^D. \quad (46)$$

In order to describe reversible dynamics, the reversible part of a current (superscript R) must have the same behavior under time reversal as the time derivative of the appropriate variable in Eqs. (33)–(39). In contrast, the dissipative parts of the currents (superscript D) have the opposite sign under time reversal as the time derivatives of the variables. According to the second law of thermodynamics, the entropy production has to vanish identically for reversible processes, $R \equiv 0$, and the entropy obeys a conservation law, Eq. (34). For

irreversible processes $R > 0$ is required. With the help of the full set of dynamic equations the Gibbs relation Eq. (18) leads to an expression for the entropy production R/T , bilinear in the currents and thermodynamic conjugates. This can be used to impose the restrictions on the reversible and irreversible parts of the currents, separately, in particular on the form of cross-coupling terms.

Since we restrict ourselves to a linear description, we do not have to worry about which velocity should be chosen for the transport derivatives [53].

C. Reversible dynamics

Implementing the condition $R = 0$ and the required behavior under time reversal and parity, we obtain the following expressions for the reversible contributions to the currents:

$$j_i^{\sigma R} = 0, \quad (47)$$

$$\sigma_{ij}^R = -\Psi_{ij} - \lambda_{kji} \nabla_l \Phi_{kl}^m + \xi_{kji}^R W_k, \quad (48)$$

$$Y_{ij}^R = -A_{ij}, \quad (49)$$

$$X_i^{mR} = -\lambda_{ijk} A_{jk}, \quad (50)$$

$$I_\varphi^R = \mu, \quad (51)$$

$$Z_i^R = -\xi_{ijk}^R A_{jk}, \quad (52)$$

where $A_{ij} = (1/2)(\nabla_i v_j^n + \nabla_j v_i^n)$.

Inspecting Eqs. (48)–(52) we see that σ_{ij}^R and I_φ^R are even under time reversal, while Y_{ij}^R , X_i^{mR} , and Z_i^R are odd under time reversal—just as expected from the general analysis discussed above for the time derivative of the variables and their time reversal properties.

The density current g_i is at the same time the momentum density and therefore cannot have dissipative contributions. It is part of the kinetic energy and has been given by Eq. (26) or (27). The result for I_φ^R , Eq. (51), follows from the fact that φ is the canonical conjugate to the particle number [39,44].

The material tensors λ_{ijk} and ξ_{ijk}^R describe the coupling of the stress tensor with the quasicurrents X_i^{mR} and Z_i^R and have the form

$$\alpha_{ijk} = \alpha(m_k^0 \delta_{ij}^\perp + m_j^0 \delta_{ik}^\perp). \quad (53)$$

These tensors have to be symmetric in the last two indices, the first index has to be transverse to m_i , and they must contain an odd number of m_i factors because m_i is a director.

We also point out that—except for the terms related to superfluidity—the reversible currents for the polar phase are isomorphic to those given in uniaxial nematic elastomers [54].

D. Irreversible dynamics and entropy production

We can use the dissipation function R as a Lyapunov functional to derive the irreversible currents and quasicurrents. This automatically includes the famous reciprocity rules for dissipative cross couplings [47]. One can expand the function R (R/T is the amount of entropy produced within a unit volume per unit time) into the thermodynamic forces using the same symmetry arguments as in the case of the energy

density. We obtain

$$\begin{aligned} R = & \frac{1}{2} \kappa_{ij} (\nabla_i T) (\nabla_j T) + \frac{1}{2} v_{ijkl} A_{ij} A_{kl} \\ & + \frac{1}{2} \xi_{ij} (\nabla_k \Psi_{ik}) (\nabla_l \Psi_{jl}) + \xi_{ij}^T (\nabla_i T) (\nabla_k \Psi_{jk}) \\ & + \zeta (\nabla_i \lambda_i^s) (\nabla_j \lambda_j^s) + \zeta_{ij}^n A_{ij} (\nabla_k \lambda_k^s) \\ & + \frac{1}{2} b \delta_{ij}^\perp (\nabla_l \Phi_{il}^m) (\nabla_m \Phi_{jm}^m) + \xi^m \delta_{ij}^\perp W_i (\nabla_l \Phi_{jl}^m) \\ & + \frac{1}{2} \tau \delta_{ij}^\perp W_i W_j + \xi_{ijk}^D (\nabla_i W_k) (\nabla_l \Psi_{jl}). \end{aligned} \quad (54)$$

The second rank tensors κ_{ij} , ξ_{ij}^T , ξ_{ij} , and ζ_{ij}^n take the form

$$\alpha_{ij} = \alpha_{\parallel} m_i^0 m_j^0 + \alpha_{\perp} \delta_{ij}^\perp. \quad (55)$$

The contributions $\sim \tau$, b , and ξ^m contain δ_{ij}^\perp , since rotations of m_i and the relative rotations are perpendicular to the preferred direction [54].

The viscosity tensor v_{ijkl} has the same form as the elastic tensor, Eq. (22), containing five viscosities $v_1 - v_5$.

The tensor ξ_{ijk}^D has to be transverse in the third index and must contain an odd number of m_i factors because m_i is a director. It takes the form

$$\xi_{ijk}^D = \xi_1^D m_i^0 \delta_{jk}^\perp + \xi_2^D m_j^0 \delta_{ik}^\perp. \quad (56)$$

To obtain the dissipative parts of the currents and quasicurrents we take the partial derivatives with respect to the appropriate thermodynamic force

$$j_i^{\sigma D} = - \left(\frac{\partial R}{\partial (\nabla_i T)} \right)_{\dots} = -\kappa_{ij} \nabla_j T - \xi_{ij}^T \nabla_k \Psi_{jk}, \quad (57)$$

$$\sigma_{ij}^D = - \left(\frac{\partial R}{\partial (\nabla_j v_i^n)} \right)_{\dots} = -v_{ijkl} A_{kl} - \zeta_{ij}^n \nabla_k \lambda_k^s, \quad (58)$$

$$\begin{aligned} Y_{ij}^D = & \left(\frac{\partial R}{\partial \Psi_{ij}} \right)_{\dots} = -\frac{1}{2} \nabla_i (\xi_{jk} \nabla_l \Psi_{kl} + \xi_{jk}^T \nabla_k T) \\ & - \frac{1}{2} \nabla_j (\xi_{ik} \nabla_l \Psi_{kl} + \xi_{ik}^T \nabla_k T) \\ & - \frac{1}{2} (\xi_{lik}^D \nabla_j + \xi_{ljk}^D \nabla_i) \nabla_l W_k, \end{aligned} \quad (59)$$

$$X_i^{mD} = \left(\frac{\partial R}{\partial \nabla_l \Phi_{il}^m} \right)_{\dots} = b \delta_{ij}^\perp \nabla_k \Phi_{jk}^m + \xi^m \delta_{ij}^\perp W_j, \quad (60)$$

$$I_\varphi^D = - \left(\frac{\partial R}{\partial (\nabla_k \lambda_k^s)} \right)_{\dots} = -\zeta \nabla_k \lambda_k^s - \zeta_{ij}^n A_{ij}, \quad (61)$$

$$\begin{aligned} Z_i^D = & \left(\frac{\partial R}{\partial W_i} \right)_{\dots} = \tau \delta_{ij}^\perp W_j + \xi^m \delta_{ij}^\perp \nabla_l \Phi_{jl}^m \\ & + \xi_{kji}^D \nabla_k \nabla_l \Psi_{jl}. \end{aligned} \quad (62)$$

Inspecting the time reversal behavior of the dissipative currents we can verify that all contributions have the opposite sign under time reversal as the corresponding time derivative of the associated variable. Looking at the heat conduction term $\sim \kappa_{ij} \nabla_j T$ and the viscous term $\sim v_{ijkl} A_{kl}$ this behavior for the heat current and the stress tensor is already familiar from the hydrodynamics of a simple fluid.

In this section we have so far taken into account relative rotations between the preferred direction for the aerogel ζ_i and the preferred direction in orbit space m_i as a macroscopic variable with a finite, but sufficiently long, relaxation time. Should

this relaxation time be very short, the relative rotations are no longer macroscopic variables, meaning $\Omega_i = 0$ macroscopically, and therefore $W_i = 0$. In this case the cross-coupling terms between the relative rotations and the other macroscopic variables no longer exist. This affects in the reversible dynamics, Eq. (48), the coupling to the stress tensor ($\xi_{kji}^R = 0$), and in the dissipative dynamics, Eq. (54), the coupling to strains ($\xi_{ijk}^D = 0$) and to gradients of the vector m_i , ($\xi^m = 0$). In the statics, $W = 0$ leads to an irrelevant renormalization of the elastic tensor μ_{ijkl} in Eq. (29).

E. On the coupling of orbit and spin space in the presence of anisotropic elasticity

In this section we have shown so far that the additional preferred direction in orbital space induced by the anisotropic aerogel leads to qualitatively new phenomena due to the coupling to strains and relative rotations. This naturally leads to the question how these results affect the spin degrees of freedom and their dynamics. In contrast to the case of the distorted B phase, which will be discussed in Sec. V, the order parameter of the polar phase in an anisotropic aerogel does not generate a preferred direction in spin space. Thus the only coupling between spin and orbital space is due to the magnetic dipole interaction the effects of which have already been discussed for the superfluid polar phase in the bulk [13,33,34] and in an isotropic aerogel [33,34]. We also emphasize that there are no coupling terms to spin space generated by the anisotropic elasticity of the aerogel. The spin dynamics of the polar phase in an anisotropic aerogel including the magnetic dipole interaction as well as external magnetic fields has been studied quite extensively over the last decade or so [1,6,7,18,19,21] and we refer to these theoretical and experimental works for further details.

Comparing our analysis of the orbital space to that of $^3\text{He-A}$ a close similarity emerges: there is flow alignment and the sound absorption of first, second, and fourth sound becomes anisotropic. These are qualitative predictions which one might be able to test experimentally for the polar phase in an anisotropic aerogel in well aligned samples. So far we are not aware of any experiments to check these results.

IV. THE ORBITAL DYNAMICS OF THE DISTORTED A PHASE IN THE PRESENCE OF AN ELASTIC AEROGEL

A. Statics and thermodynamics

The relevant variables to describe the orbital dynamics of the distorted A phase have been discussed in Sec. II. Using Eqs. (1), (2), (6), (11), and (17) we get the Gibbs relation

$$d\varepsilon = Td\sigma + \mu d\rho + v_i^n dg_i + \lambda_i^s dv_i^s + \Phi_{ij}^l d(\nabla_j l_i) + \Psi_{ij} d\epsilon_{ij} + Wd\Omega. \quad (63)$$

Compared to the polar phase in an aerogel we notice three general differences: First, the symmetry variable describing rotation of the preferred axis in orbital space δl_i is odd under time reversal, in contrast to the polar phase, where δm_i is even. Second, in the distorted A phase there is a second preferred axis ζ_i^0 orthogonal to l_i . It is not a symmetry variable, but given by the anisotropy of the aerogel. Its main effect is making the system biaxial (of orthorhombic symmetry). The

variable Ω , describing relative rotations between l_i and ζ_i^0 , is a single-component quantity that is odd under time reversal, while in the polar phase the corresponding relative rotations have two components and are even under time reversal. Finally, there is a third type of velocity $(\hbar/2m)(\nabla \times \mathbf{l})$, specific for superfluid $^3\text{He-A}$ and first introduced by Graham [49]. For the full hydrodynamic equations of the usual A phase we refer to Refs. [35,43,49,50]

For the energy density we get

$$\begin{aligned} \varepsilon = & \frac{1}{2}\rho_0 \left(\frac{\rho^s}{\rho^n}\right)_{ij} v_i^s v_j^s + \frac{1}{2} \left(\frac{1}{\rho^n}\right)_{ij} g_i g_j \\ & - \left(\frac{\rho^s}{\rho^n}\right)_{ij} v_i^s g_j - \left(\frac{C}{\rho^n}\right)_{ij} (\nabla \times \mathbf{l})_i g_j \\ & + \frac{1}{2} \mu_{ijkl} \epsilon_{ij} \epsilon_{kl} + \frac{1}{2} K_{ijkl} (\nabla_j l_i) (\nabla_l l_k) \\ & + \frac{1}{2} D_1 \Omega^2 + D_2 (l_i^0 \zeta_i^0 + l_j^0 \zeta_j^0) \Omega \epsilon_{ij} \\ & + \sigma_{ijk}^\sigma (\nabla_j l_i) (\nabla_k \sigma) + \sigma_{ijk}^\rho (\nabla_j l_i) (\nabla_k \rho) \\ & + \epsilon_{ij} (\chi_{ij}^\sigma \delta \sigma + \chi_{ij}^\rho \delta \rho) + c_{\rho\rho} (\delta \rho)^2 + c_{\sigma\sigma} (\delta \sigma)^2 \\ & + c_{\rho\sigma} (\delta \rho) (\delta \sigma), \end{aligned} \quad (64)$$

where the second rank tensors have the biaxial form, e.g.,

$$\left(\frac{C}{\rho^n}\right)_{ij} = \frac{C_{\parallel}}{\rho_{\parallel}^n} l_i^0 l_j^0 + \frac{C_2}{\rho_{\perp}^n} \zeta_i^0 \zeta_j^0 + \frac{C_{\perp}}{\rho_{\perp}^n} \delta_{ij}^{\perp}, \quad (65)$$

where, in the present section, δ_{ij}^{\perp} is the biaxial transverse Kronecker symbol

$$\delta_{ij}^{\perp} = \delta_{ij} - l_i^0 l_j^0 - \zeta_i^0 \zeta_j^0. \quad (66)$$

The orthorhombic biaxial tensor of the elastic energy contains nine elastic moduli and can be found, e.g., in Ref. [55]. The tensor K_{ijkl} is the biaxial version of the nematic Frank elastic tensor for one director and reads

$$\begin{aligned} K_{ijkl} = & K_{11} \zeta_i^0 \zeta_j^0 \zeta_k^0 \zeta_l^0 + K_{12} (\zeta_i^0 \zeta_j^0 \delta_{kl}^{\perp} + \delta_{ij}^{\perp} \zeta_k^0 \zeta_l^0) \\ & + K_{14} \delta_{ij}^{\perp} \delta_{kl}^{\perp} + K_2 l_p^0 \epsilon_{pij} l_q^0 \epsilon_{qkl} \\ & + K_{31} l_j^0 l_l^0 \zeta_i^0 \zeta_k^0 + K_{32} l_j^0 l_l^0 \delta_{ik}^{\perp}. \end{aligned} \quad (67)$$

Relative rotations couple via the D_2 term to shear strains in the \mathbf{l}^0/ζ^0 plane. Couplings of strains to gradients of the scalar variables are provided by the tensors χ_{ij}^σ and χ_{ij}^ρ , which are of the form Eq. (65). The third rank tensors are given by

$$\sigma_{ijk}^{\xi} = \sigma_1^{\xi} (l_j^0 \delta_{ik}^{\perp} + l_k^0 \delta_{ij}^{\perp}) + \sigma_2^{\xi} (l_j^0 \zeta_i^0 \zeta_k^0 + l_k^0 \zeta_i^0 \zeta_j^0), \quad (68)$$

where ξ can be either ρ or c .

The explicit expressions for the conjugated variables in terms of the hydrodynamic and macroscopic variables are

$$g_i = \rho_{ij}^n v_j^n + \rho_{ij}^s v_j^s + C_{ij} (\nabla \times \mathbf{l})_j, \quad (69)$$

$$\lambda_i^s = \left(\frac{\partial \varepsilon}{\partial v_i^s}\right)_{\dots} = g_i - \rho_0 v_i^n - \rho_0 \left(\frac{C}{\rho^n}\right)_{ij} (\nabla \times \mathbf{l})_j, \quad (70)$$

$$\begin{aligned} \Phi_{ij}^l = & \left(\frac{\partial \varepsilon}{\partial (\nabla_j l_i)}\right)_{\dots} = K_{ijkl} (\nabla_l l_k) + \sigma_{ijk}^\sigma (\nabla_k \sigma) \\ & + \sigma_{ijk}^\rho (\nabla_k \rho) + \delta_{ik}^{\perp} \epsilon_{kpj} \left(\frac{C}{\rho^n}\right)_{pq} g_q, \end{aligned} \quad (71)$$

$$\Psi_{ij} = \left(\frac{\partial \varepsilon}{\partial \varepsilon_{ij}} \right)_{\dots} = \mu_{ijkl} \varepsilon_{kl} + \chi_{ij}^{\sigma} \delta \sigma + \chi_{ij}^{\rho} \delta \rho + \varphi_3^R (\varepsilon_{ikp} \delta_{jl}^{\perp} + \varepsilon_{jlp} \delta_{ik}^{\perp} + \varepsilon_{ilp} \delta_{jk}^{\perp} + \varepsilon_{jkp} \delta_{il}^{\perp}) l_p^0, \quad (85)$$

$$+ D_2 (l_j^0 \zeta_i^0 + l_i^0 \zeta_j^0) \Omega, \quad (72)$$

$$W = \left(\frac{\partial \varepsilon}{\partial \Omega} \right)_{\dots} = D_1 \Omega + D_2 (l_j^0 \zeta_i^0 + l_i^0 \zeta_j^0) \varepsilon_{ij}, \quad (73)$$

$$\delta T = \left(\frac{\partial \varepsilon}{\partial \delta \sigma} \right)_{\dots} = \chi_{ij}^{\sigma} \varepsilon_{ij} + 2c_{\sigma\sigma} \delta \sigma + c_{\rho\sigma} \delta \rho - \sigma_{ijk}^{\sigma} \nabla_j \nabla_k l_i, \quad (74)$$

$$\delta \mu = \left(\frac{\partial \varepsilon}{\partial \delta \rho} \right)_{\dots} = \chi_{ij}^{\rho} \varepsilon_{ij} + 2c_{\rho\rho} \delta \rho + c_{\rho\sigma} \delta \sigma - \sigma_{ijk}^{\rho} \nabla_j \nabla_k l_i. \quad (75)$$

We note that the Galilean requirement $\rho_0 \delta_{ij} = \rho_{ij}^n + \rho_{ij}^s$ is still fulfilled, since $(\nabla \times \mathbf{l})$ is Galilei invariant.

B. Dynamic equations and reversible dynamics

The dynamic Eqs. (33)–(39) of the polar phase can be taken over for the distorted A phase with two exceptions. The dynamic equation for m_i , Eq. (36), has to be replaced by a dynamic equation for l_i ,

$$\dot{l}_i + \varepsilon_{ijk} l_j^0 \omega_k + X_i^l = 0, \quad (76)$$

with $\omega_i = (1/2) \varepsilon_{ijk} \nabla_j v_k^n$ the vorticity. For the dynamics of the relative rotation variable Ω , we replace Eq. (39) by

$$\dot{\Omega} + Z = 0. \quad (77)$$

For the reversible currents up to linear order in the thermodynamic forces we find

$$j_i^{\sigma R} = -\kappa_{ij}^R \nabla_j T + \xi_{ij}^{TR} \nabla_l \Psi_{jl}, \quad (78)$$

$$\sigma_{ij}^R = -\Psi_{ij} - \frac{1}{2} \lambda_{kji} \nabla_l \Phi_{kl}^l - \varphi_{ijkl}^R A_{kl} + \xi_{ij}^{\sigma R} W - \lambda_{ij}^L \nabla_k \lambda_k^s, \quad (79)$$

$$Y_{ij}^R = -A_{ij} - \frac{1}{2} (\bar{\varphi}_{ik}^R \nabla_j + \bar{\varphi}_{jk}^R \nabla_i) \nabla_p \Psi_{kp} - \frac{1}{2} (\xi_{jk}^{TR} \nabla_i + \xi_{ik}^{TR} \nabla_j) \nabla_k T, \quad (80)$$

$$X_i^{lR} = -\beta_{ij}^R \nabla_k \Phi_{jk}^l - \lambda_{ijk} A_{kj} + \xi_i^{XR} W, \quad (81)$$

$$I_{\varphi}^R = \mu - \lambda_{ij}^L A_{ij}, \quad (82)$$

$$Z^R = \xi_j^{XR} \nabla_l \Phi_{jl}^l - \xi_{ij}^{\sigma R} A_{ij}, \quad (83)$$

with $A_{ij} = (1/2) (\nabla_i v_j^n + \nabla_j v_i^n)$. Note that g_i is already given in Eqs. (69) and (70).

There are self-coupling terms characterized by antisymmetric material tensors, in particular $\kappa_{ij} - \kappa_{ji}$, $\beta_{ij} - \beta_{ji}$, $\bar{\varphi}_{ik}^R = -\bar{\varphi}_{ki}^R$, and $\varphi_{ijkl}^R = -\varphi_{klij}^R$, thus vanishing in the entropy production. They have to contain an odd number of l_i factors to give the correct time reversal behavior $a^R \in \{\alpha^R, \beta^R, \bar{\varphi}^R\}$,

$$a_{ij}^R = a^R \varepsilon_{ijk} l_k^0, \quad (84)$$

$$\varphi_{ijkl}^R = \varphi_1^R (\varepsilon_{ikp} l_j^0 l_l^0 + \varepsilon_{jlp} l_i^0 l_k^0 + \varepsilon_{ilp} l_j^0 l_k^0 + \varepsilon_{jkp} l_i^0 l_l^0) l_p^0 + \varphi_2^R (\varepsilon_{ikp} \zeta_j^0 \zeta_l^0 + \varepsilon_{jlp} \zeta_i^0 \zeta_k^0 + \varepsilon_{ilp} \zeta_j^0 \zeta_k^0 + \varepsilon_{jkp} \zeta_i^0 \zeta_l^0) l_p^0$$

For $\varphi_2 = \varphi_3$ the uniaxial form for ${}^3\text{He-A}$ [50] is found.

There are new reversible cross couplings between relative rotations and normal flow $\xi_{ij}^{\sigma R}$ and elasticity ξ_{ij}^{XR} , and between superfluid and normal flow λ_{ij}^L . They all contain one l_i factor, require biaxiality, and are of the symmetric form

$$E_{ij} = E (l_i^0 \zeta_j^0 + l_j^0 \zeta_i^0), \quad (86)$$

with $E \in \{\xi^{\sigma R}, \lambda^L\}$.

The third rank tensor λ_{ijk} , called flow alignment tensor in the nematic liquid crystal context, has to be symmetric in j, k , odd under time reversal, and reads in the biaxial case

$$\lambda_{ijk} = \lambda_1 (\delta_{ij}^{\perp} l_k^0 + \delta_{ik}^{\perp} l_j^0) + \lambda_2 (\zeta_i^0 \zeta_j^0 l_k^0 + \zeta_k^0 \zeta_i^0 l_j^0). \quad (87)$$

For $\lambda_1 = \lambda_2$ the uniaxial form known from ${}^3\text{He-A}$ [49] is regained.

There is a first rank material tensor ξ_i^{XR} , odd under time reversal, relating relative relations to rotations of l_i that exists only in a biaxial state

$$\xi_i^{XR} = \xi^{XR} \varepsilon_{ijk} l_j^0 \zeta_k^0 \quad (88)$$

and is a new term and neither exists in nematic liquid crystalline elastomers [54] nor in superfluid ${}^3\text{He-A}$.

C. Irreversible dynamics and entropy production

The dissipation function reads

$$R = \frac{1}{2} \kappa_{ij} (\nabla_i T) (\nabla_j T) + \frac{1}{2} v_{ijkl} A_{ij} A_{kl} + \frac{1}{2} \xi_{ij} (\nabla_k \Psi_{ik}) (\nabla_l \Psi_{jl}) + \xi_{ij}^T (\nabla_i T) (\nabla_k \Psi_{jk}) + \frac{1}{2} \zeta (\nabla_i \lambda_i^s) (\nabla_j \lambda_j^s) + \zeta_{ij}^n (\nabla_k \lambda_k^s) A_{ij} + \frac{1}{2} b_{ij} (\nabla_l \Phi_{il}^l) (\nabla_m \Phi_{jm}^l) + \xi^m \zeta_i^0 W (\nabla_l \Phi_{il}^l) + \frac{1}{2} \tau W^2 + \xi_{ij}^D (\nabla_i W) (\nabla_l \Psi_{jl}), \quad (89)$$

where the second rank tensors κ_{ij} , ζ_{ij}^n , ξ_{ij} , and ξ_{ij}^T are of the biaxial form with three coefficients, Eq. (65), and the viscosity tensor v_{ijkl} is of the orthorhombic biaxial form with nine viscosities [55]. The tensor

$$b_{ij} = b_2 \zeta_i^0 \zeta_j^0 + b_{\perp} \delta_{ij}^{\perp} \quad (90)$$

is transverse to l_i , and finally the tensor ξ_{ij}^D contains an odd number of l_i and has the form of Eq. (86).

To obtain the dissipative parts of the currents and quasi-currents we take the partial derivatives with respect to the appropriate thermodynamic force

$$j_i^{\sigma D} = - \left(\frac{\partial R}{\partial (\nabla_i T)} \right)_{\dots} = -\kappa_{ij} \nabla_j T - \xi_{ij}^T \nabla_k \Psi_{jk}, \quad (91)$$

$$\sigma_{ij}^D = - \left(\frac{\partial R}{\partial A_{ij}} \right)_{\dots} = -v_{ijkl} A_{kl} - \zeta_{ij}^n \nabla_k \lambda_k^s, \quad (92)$$

$$Y_{ij}^D = \left(\frac{\partial R}{\partial \Psi_{ij}} \right)_{\dots} = -(\xi_{jk} \nabla_i + \xi_{ik} \nabla_j) \nabla_l \Psi_{kl} - (\xi_{jk}^T \nabla_i + \xi_{ik}^T \nabla_j) \nabla_k T - (\xi_{jk}^D \nabla_i + \xi_{ik}^D \nabla_j) \nabla_k W, \quad (93)$$

$$I_\varphi^D = -\left(\frac{\partial R}{\partial(\nabla_k \lambda_k^s)}\right)_{\dots} = -\zeta \nabla_k \lambda_k^s - \zeta_{ij}^n A_{ij}, \quad (94)$$

$$Z^D = \left(\frac{\partial R}{\partial W}\right)_{\dots} = \tau W + \xi^m \zeta_i^0 \nabla_k \Phi_{ik}^l - \xi_{ij}^D \nabla_i \nabla_k \Psi_{jk}, \quad (95)$$

$$X_i^{lD} = \left(\frac{\partial R}{\partial(\nabla_k \Phi_{ik}^l)}\right)_{\dots} = b_{ij} \nabla_k \Phi_{jk}^l + \xi^m \zeta_i^0 W. \quad (96)$$

D. Spin-orbit coupling and the preferred directions in spin space

Neglecting dipole-dipole interaction spin space and orbit space are independent. That means vectorial quantities in orbit space (Latin indices) cannot couple to vectorial quantities in spin space (Greek indices). The scalar variable Ω , in principle, could couple to variables in both spaces. Ω is odd under time reversal and can couple to orbit space, since there, l_i^0 with the same time reversal properties exists. The preferred direction in spin space d_v^0 is even under time reversal and Ω cannot couple.

The weak dipole-dipole interaction registers spin and orbit space such that $d^0 \parallel l^0$, rendering the distinction between Greek and Latin indices obsolete. Since the preferred direction of the aerogel ζ_i^0 is perpendicular to l_i^0 , it is now also perpendicular to d_i^0 and spin space becomes biaxial. This is a situation found in the undistorted A phase only after having applied a (strong) magnetic field [51]. Although the preferred direction of the aerogel enters the spin space dynamics, when the spin-orbit coupling is taken into account, the elastic deformations of the aerogel do not.

Thus we have shown in this section that the hydrodynamic variables associated with the superfluid order parameter in the distorted A phase in a uniaxial aerogel are the same as in the usual A phase: the superfluid velocity v_i^s , the deviations from the preferred direction in real space \hat{l} , δl_i , and the deviations from the preferred direction in spin space δd_v . In orbit space the macroscopic dynamics of the A phase and the distorted A phase without an aerogel are isomorphic. The uniaxial aerogel brings along as macroscopic variables the strain field ϵ_{ij} and the variable Ω describing relative rotations between l_i and ζ_i . In addition, the distorted A phase in an aerogel acquires biaxiality in real space due to the preferred direction of the aerogel in orbital space ζ_i . This biaxiality makes possible a reversible coupling between relative rotations (between the \hat{l} vector and the preferred direction of the uniaxial aerogel) and the \hat{l} dynamics. It will be undoubtedly a challenge to detect this coupling experimentally. Finally, the dipole interaction leads also to a biaxiality in spin space.

E. Uniaxiality in disorder-generated distorted A phases

An important point to discuss for the distorted A phase is the issue over which length scales this phase can be biaxial in case when there is disorder over large length scales in the direction perpendicular to the preferred direction of the anisotropic aerogel ζ_i . This issue has already been addressed in a number of papers including Refs. [1,11] and it has been pointed out that in many cases the biaxiality is averaged out over length scales large compared to $1 \mu\text{m}$.

On the other hand, it has been emphasized quite recently [9] that in squeezed aerogels of “mullites” the 2D LIM states become strongly anisotropic in the direction perpendicular to

the preferred direction ζ_i and a biaxial system prevails over large length scales [9]. It is this latter state which we have discussed in detail in this section.

The macroscopically uniaxial case is obtained from our general expressions above, by discarding the preferred direction l_i^0 . This reduces the number of independent coefficients in various material parameters, e.g., in K_{ijkl} from 6 to 3, in symmetric second rank tensors of the form of Eq. (65) from 3 to 2, in b_{ij} and λ_{ijk} from 2 to 1, and some couplings vanish completely, in particular those involving relative rotations (including ξ_i^{XR} , $\xi^{\sigma R}$, ξ_{ij}^D , and ξ_m) as well as λ^L , the coupling between normal and superfluid velocity. As a result, the uniaxial orbital dynamics is isomorphic to that of the polar phase in the presence of an aerogel.

V. HYDRODYNAMICS OF THE DISTORTED B PHASE IN THE PRESENCE OF AN ELASTIC AEROGEL

This section is split into two parts. Section V A discusses the orbital dynamics and Sec. V B the spin part as well as the coupling to the orbital part. For the full hydrodynamic equations of usual $^3\text{He}-B$ we refer to Refs. [43,52].

A. The orbital part

Neglecting spin-orbit coupling we can set up the hydrodynamics of the orbital variables separately. According to the discussion in Sec. II, Eqs. (1), (2), (6), and (13), we get the Gibbs relation

$$d\varepsilon = T d\sigma + \mu d\rho + v_i^n d g_i + \lambda_i^s d v_i^s + \Psi_{vjk}^n d \nabla_j n_{vk} + \Psi_{ij} d \epsilon_{ij} \quad (97)$$

relating the variables with the conjugate quantities.

The rotation matrix n_{vk} , the symmetry variable of the $^3\text{He}-B$ phase, is not a purely orbital variable, but also lives in spin space. It turned out for $^3\text{He}-B$ that it is impossible to construct any purely orbital material tensor out of n_{vk} [52], with the result that no couplings to the orbital variables are possible. This remains true for the distorted B phase, since couplings between n_{vk} and the strains ϵ_{ij} are impossible as well. Thus, we are left with the variables of a simple fluid, the superfluid velocity and the elastic strains. However, the distorted B phase has a preferred direction (in orbital space) ζ_i^0 , due to the anisotropy of the aerogel strands. This is not a hydrodynamic variable, but makes hydrodynamics of the distorted B phase anisotropic, in marked contrast to that of the (undistorted) B phase.

The orbital dynamics of the distorted B phase can be deduced from that of polar phase in the presence of an aerogel, Sec. III, by removing the symmetry variable m_i and the relative rotation variable Ω_i and replacing the preferred direction m_i^0 by ζ_i^0 . We simply list here the relevant equations for the statics:

$$g_i = \rho_{ij}^n v_j^n + \rho_{ij}^s v_j^s, \quad (98)$$

$$\lambda_i^s = g_i - \rho_0 v_i^n, \quad (99)$$

$$\Psi_{ij} = \mu_{ijkl} \epsilon_{kl} + \chi_{ij}^\sigma \delta\sigma + \chi_{ij}^\rho \delta\rho, \quad (100)$$

$$\delta T = \chi_{ij}^\sigma \epsilon_{ij} + 2c_{\sigma\sigma} \delta\sigma + c_{\rho\sigma} \delta\rho, \quad (101)$$

$$\delta\mu = \chi_{ij}^\rho \epsilon_{ij} + 2c_{\rho\rho} \delta\rho + c_{\rho\sigma} \delta\sigma, \quad (102)$$

and the dynamics

$$j_i^\sigma = -\kappa_{ij} \nabla_j T - \xi_{ij}^T \nabla_k \Psi_{jk}, \quad (103)$$

$$\sigma_{ij} = -\Psi_{ij} - v_{ijkl} A_{kl} - \zeta_{ij}^n \nabla_k \lambda_k^s, \quad (104)$$

$$I_\varphi = \mu - \zeta \nabla_k \lambda_k^s - \zeta_{ij}^n A_{ij}, \quad (105)$$

$$Y_{ij} = -A_{ij} - \frac{1}{2} \nabla_i (\xi_{jk} \nabla_l \Psi_{kl} + \xi_{jk}^T \nabla_k T) \\ - \frac{1}{2} \nabla_j (\xi_{ik} \nabla_l \Psi_{kl} + \xi_{ik}^T \nabla_k T), \quad (106)$$

where the currents j_i^σ , σ_{ij} , I_φ , and Y_{ij} are defined by the dynamic equations Eqs. (34), (35), (37), and (38), respectively. Material tensors of rank-2 and of rank-4 are given by Eqs. (21) and (22), respectively, with m_i^0 replaced by ζ_i^0 and $\delta_{ij}^{\frac{1}{2}} = \delta_{ij} - \zeta_i^0 \zeta_j^0$.

The symmetry variable n_{vk} shows up in the spin dynamics of ${}^3\text{He-B}$ [52], and of course, in the spin dynamics of the distorted B phase as well, see below.

B. The spin part

In superfluid ${}^3\text{He-B}$ the spin hydrodynamics is described by the rotation matrix n_{vi} , describing the relative orientation between spin and orbital space. Neglecting spin-orbit coupling effects, n_{vi} does not couple to the orbital variables, but in spin space to the dynamics of the magnetization s_v [52]. This is also applicable to the distorted B phase, where in addition, the aerogel strands give rise to a preferred direction ζ_i^0 . The latter is manifest in spin space as $\zeta_v^0 = n_{vi} \zeta_i^0$. Thus, in contrast to the B phase in the bulk, the distorted B phase in a uniaxial aerogel also acquires a preferred direction in spin space, which profoundly influences the spin wave spectrum as we will show in the following.

Using the appropriate Gibbs relation, Eqs. (3) and (13),

$$d\varepsilon = \Psi_{vjk}^n d\nabla_j n_{vk} + h_v ds_v \quad (107)$$

the conjugate quantities are related to the variables by

$$h_v = \left[(1/\chi_{\parallel}) \zeta_v^0 \zeta_\mu^0 + (1/\chi_{\perp}) \delta_{v\mu} \right] s_\mu, \quad (108)$$

$$\Psi_{vik}^n = M_{v\mu i j k l} \nabla_j n_{\mu l}, \quad (109)$$

with

$$M_{v\mu i j k l} = M_1 \delta_{v\mu} \delta_{ij} \delta_{kl} + M_3 \zeta_v^0 \zeta_\mu^0 \delta_{ij} \delta_{kl} \\ + M_4 \delta_{v\mu} \zeta_i^0 \zeta_j^0 \delta_{kl} + M_5 \zeta_v^0 \zeta_\mu^0 \zeta_i^0 \zeta_j^0 \delta_{kl} \\ + M_2 (\epsilon_{kip} \epsilon_{jlt} + \epsilon_{kjp} \epsilon_{ilt}) n_{vp}^0 n_{\mu t}^0, \quad (110)$$

where M_1 and M_2 are already present in the B phase.

The dynamic equations are

$$\dot{s}_v + \nabla_k J_{vk} = 0, \quad (111)$$

$$\dot{n}_{vi} + Z_{vi} = 0. \quad (112)$$

According to general spin dynamics the reversible parts of the currents are [52]

$$J_{vi}^R = \gamma \epsilon_{v\mu\lambda} n_{\lambda j}^0 \Psi_{\mu ij}^n, \quad (113)$$

$$Z_{vi}^R = \gamma \epsilon_{v\mu\lambda} n_{\mu i}^0 h_\lambda, \quad (114)$$

with γ the gyromagnetic ratio.

The dissipative dynamics involves higher order gradient terms, $J_{vi}^D \sim \nabla_k h_\mu$ and $Z_{vi}^D \sim \nabla_j \Psi_{\mu jk}^n$, and will not be discussed here in detail.

In the (isotropic) B phase the spin dynamics consists of three propagating phononlike excitations with frequency ω and wave vector \mathbf{k} , reflecting the spontaneously broken rotational symmetry due to n_{vi} . There is a longitudinal spin wave involving $\nabla \cdot \mathbf{s}$, and two degenerate transverse ones involving $\nabla \times \mathbf{s}$, with $\omega_l^2 = c_l^2 k^2$ and $\omega_{tr}^2 = c_{tr}^2 k^2$, respectively.

In the distorted B phase there are still three phononlike modes, but in contrast to the B phase the mode velocities are anisotropic. We describe this anisotropy by

$$k_{\parallel}^2 \equiv -\zeta_i^0 \zeta_j^0 \nabla_i \nabla_j = k^2 \cos^2 \Theta, \quad (115)$$

with Θ the angle between the wave vector \mathbf{k} and the preferred direction ζ^0 .

In addition, two of the three modes are coupled. Such a scenario is found in the B phase only after the spin-orbit coupling is taken into account [52]. In particular, we find in the distorted B phase for the single mode involving transverse excitations ($\epsilon_{v\mu\lambda} \nabla_\mu s_\lambda$)

$$\omega_1^2 = c_1^2(\Theta) k^2 \\ = \frac{\gamma^2}{\chi_{\perp}} k^2 (2M_1 + M_3 + [2M_4 + M_5] \cos^2 \Theta), \quad (116)$$

where c_1 reduces to c_{tr} in the isotropic limit.

The dissipative part of the dynamics, not shown above, contributes diffusion to the mode spectrum in order k^2 , e.g., $\omega_1 = \pm c_1(\Theta) k + (i/2) D_1(\Theta) k^2$.

For the coupled spin wave branches, the frequencies $\omega_{2,3}$ of the longitudinal one (involving $\nabla_i s_i$) and a transverse one (involving $\zeta_i s_i$), are the solutions of the quadratic Eq. (117),

$$(\omega_{2,3}^2 - Ak^2)(\omega_{2,3}^2 - Fk^2) + Ck^4 \cos^2 \Theta = 0, \quad (117)$$

with

$$A = 2\gamma^2 \left(\frac{1}{\chi_{\perp}} + \frac{1}{\chi_{\parallel}} \right) (M_1 + M_4 \cos^2 \Theta) + \frac{8\gamma^2}{\chi_{\parallel}} M_2, \quad (118)$$

$$F = \frac{\gamma^2}{\chi_{\perp}} (2M_1 + M_3 + 8M_2 + [2M_4 + M_5] \cos^2 \Theta), \quad (119)$$

$$C = \frac{8\gamma^4}{\chi_{\perp}^2} M_2 \left(8M_2 + M_3 + M_5 \cos^2 \Theta \right. \\ \left. + \frac{2\chi_{\perp}}{\chi_{\parallel}} [M_1 - M_4 \cos^2 \Theta] \right). \quad (120)$$

In the isotropic limit, where the C contribution drops out of Eq. (117), we get $A \rightarrow c_{tr}^2$ and $F \rightarrow c_l^2$.

Taking into account spin-orbit coupling in the (isotropic) B phase, spin space and orbit space are rotated against each other by a fixed angle $\vartheta_0 = \cos^{-1}(1/4)$ about an axis d_i [13].

The orientation of d_i is arbitrary giving rise to still two spontaneously broken rotational symmetries. There is now no need to use different types of indices anymore and the rotation matrix can be written as

$$n_{ij} = \cos \vartheta (\delta_{ij} - d_i d_j) + d_i d_j + \sin \vartheta \epsilon_{ijk} d_k. \quad (121)$$

The spin-orbit energy for deviations from the equilibrium state

$$\varepsilon_{so} = \frac{B}{2} (\vartheta - \vartheta_0)^2 = \frac{B}{8 \sin^2 \vartheta_0} (n_{ii} - n_{ii}^0)^2 \quad (122)$$

is rather small. It is customary to derive spin hydrodynamics under the full threefold broken symmetry, and add ε_{so} at the end [52]. This results in a gap in the longitudinal spin wave branch that is manifest in NMR as the longitudinal shift. In addition, this mode is coupled to one of the transverse modes.

In the distorted B phase spin-orbit coupling also fixes the rotation angle to ϑ_0 with similar consequences to the mode spectrum. In particular, the mode longitudinal to the preferred direction, $\zeta_i s_i$, acquires a gap (in the limit $k \rightarrow 0$)

$$\omega_2^2(k \rightarrow 0) = \gamma^2 \left(\frac{1}{\chi_{\parallel}} + \frac{1}{\chi_{\perp}} \right) B \equiv \tilde{B}. \quad (123)$$

Note that this mode is already coupled to one of the transverse modes, even without spin-orbit coupling.

Another consequence of the existence of the preferred direction in equilibrium $d_i^0 = \zeta_i^0$ is the finite energy for deviations $\delta d_i = d_i - d_i^0$,

$$\varepsilon_{sod} = \frac{D}{2} (\delta d_i)^2 = \frac{D}{8 \sin^2 \vartheta_0} [\zeta_p (n_{pi} - n_{ip})]^2, \quad (124)$$

with $D > 0$. Note that D is directly related to $\Delta_{\parallel} - \Delta_{\perp}$ of Ref. [12]. This energy gives rise to gaps in the transverse spin modes ($\epsilon_{ijk} \zeta_j s_k$) of the form

$$\omega_{1,3}^2(k \rightarrow 0) = \frac{\gamma^2}{\chi_{\perp}} \frac{1 + \cos \vartheta_0}{2 \sin^2 \vartheta_0} D \equiv \tilde{D}. \quad (125)$$

There is no spontaneously broken rotational symmetry left and all modes acquire a gap. In addition, all three spin wave modes are coupled. Such a situation is found in the B phase only after having applied an external magnetic field. In the distorted B phase an external field H_i will not induce qualitatively new features to the spin wave modes. In particular, the gaps of the modes are then given, for a field parallel to ζ_i , by $\omega_1^2 = \tilde{D}$ and $\omega_3^2 = \tilde{D} + \omega_L^2$ and, for a transverse field by $\omega_2^2 = \tilde{B} + \omega_L^2$, where $\omega_L = \gamma H$ is the Larmor frequency.

Although the preferred direction of the aerogel enters the spin space dynamics, when the spin-orbit coupling is taken into account, the elastic deformations of the aerogel do not.

In concluding this section we briefly give a comparison between the macroscopic dynamics of the usual B phase without an aerogel and the distorted B phase occurring in a uniaxially anisotropic aerogel. The hydrodynamic variables brought along by the superfluid order parameter are the same in both cases: the superfluid velocity v_i^s and three angles associated with spontaneously broken spin-orbit symmetry [13,15]. A preferred direction in orbital space is generated by the preferred direction in orbit space due to the uniaxial aerogel. Naturally this renders the hydrodynamics in orbit space to be uniaxial in contrast to the hydrodynamic of the

usual B phase. This leads in particular to an anisotropy of sound absorption of first, second, and fourth sound. Via the rotation matrix part of the order parameter a preferred direction in spin space is generated as well in the distorted B phase in a uniaxial aerogel. This leads to significant changes in the spin wave spectrum compared to the usual B phase. First of all, all spin wave velocities become anisotropic. Second, the coupling structure between the three pairs of spin waves changes considerably. In the usual B one has two degenerate pairs of transverse spin waves and one longitudinal pair. Only after taking into account the spin-orbit interaction a coupling between the different dispersion branches of spin waves arises. This is qualitatively different in the distorted B phase in a uniaxial aerogel in which case one pair of transverse spin waves becomes coupled to the longitudinal pair. Taking into account the spin-orbit interaction in this case leads to a coupling of all three pairs of spin waves, which also all acquire a gap in the long wavelength limit ($k \rightarrow 0$).

VI. SUMMARY AND PERSPECTIVE

In this paper we have studied the macroscopic behavior of the three superfluid phases of ^3He observed experimentally in anisotropic aerogels: the polar phase and the distorted A and B phases. Our focus was mainly on the macroscopic properties of the orbital part and the differences between the three phases. While all three phases share the strain as a macroscopic variable, there are significant differences, mainly because of the number of preferred directions in orbital space and their behavior under time reversal.

In the polar phase the preferred direction in orbital space is even under time reversal and one has two variables associated with relative rotations between the network and the preferred direction associated with the order parameter of the polar phase. The dynamics of this phase shares many properties with nematic liquid crystalline elastomers as long as the superfluid aspects are not considered. Except for the magnetic dipole interaction there is no coupling to spin degrees of freedom.

In the distorted A phase one has a preferred direction coming from the order parameter, which is odd under time reversal and a second preferred direction, perpendicular to the first one, coming from the anisotropic aerogel. Correspondingly, one finds numerous additional static as well as reversible and dissipative dynamic cross-coupling terms to the strain tensor and relative rotations. Due to the structure of the order parameter, there is only one variable in the distorted A phase associated with relative rotations. Without the presence of an aerogel the orbital part of the hydrodynamics of the distorted A and the A phase turn out to be isomorphic. Similar to the polar phase in an anisotropic aerogel, only the magnetic dipole interaction couples to spin degrees of freedom.

The distorted B phase in an anisotropic aerogel turns out to have macroscopic properties, which are quite different from those of the bulk B phase on one hand and from the polar and the distorted A phases in an anisotropic aerogel on the other. Due to the preferred direction of the anisotropic aerogel, the distorted B phase acquires a preferred direction and is thus anisotropic in orbit space in contrast to the superfluid B phase.

Since there is only one preferred direction in distorted B , relative rotations cannot arise as macroscopic variables. But due to the structure of the order parameter, which is closely related to a rotation matrix, and the fact that one has a preferred direction in orbit space, one can construct in a straightforward way a preferred direction in spin space as well—even in the absence of the magnetic dipole interaction and external magnetic fields. There are two direct consequences for the macroscopic behavior in spin space: all excitations become anisotropic in orbital space and longitudinal and transverse spin waves couple—even in the absence of magnetic dipole interactions and magnetic fields.

We note that all macroscopic equations for the distorted A as well as for the distorted B phase reduce in the appropriate limit (no uniaxial aerogel present and in the proper limit of the superfluid order parameter discussed in Sec. II) to the hydrodynamic equations for superfluid ^3He - A and superfluid ^3He - B [35,43,49,50,52].

The hydrodynamic description of two of the three superfluid phases discussed here (the polar phase and the distorted B phase) is uniaxial in orbital space, while the distorted A phase is biaxial. Uniaxiality has already been found for all three superfluid phases of ^3He in the bulk (A , B , and A_1) [40,45]. Thus, apart from $^3\text{P}_2$ neutron star matter [56], we have found here a candidate for showing superfluid biaxial behavior in orbital space. In Ref. [56] it has been shown that the orbit part of hydrodynamics of $^3\text{P}_2$ neutron star matter is that of a superfluid biaxial nematic liquid crystal. For further details on the macroscopic behavior of biaxial nematic liquid crystals we refer to Refs. [57–59].

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APPENDIX: TRANSIENT NETWORK

If the elastic network is transient rather than permanent, its dynamics changes from diffusive to relaxative. This happens, when, e.g., the bonding between different strands is not chemical but merely physical. In that case the thermodynamic force is the elastic stress Ψ_{ij} , rather than its divergence $\nabla_j \Psi_{ij}$. This results in various cross-coupling effects different from the permanent elastic case. Below we discuss this for the different phases, separately.

1. Polar phase

We find for the dissipation function R ,

$$\begin{aligned} R = & \frac{1}{2} \kappa_{ij} (\nabla_i T) (\nabla_j T) + \frac{1}{2} v_{ijkl} A_{ij} A_{kl} \\ & + \frac{1}{2} \xi_{ijkl} \Psi_{ij} \Psi_{jk} + \zeta (\nabla_i \lambda_i^s) (\nabla_j \lambda_j^s) \\ & + \frac{1}{2} b \delta_{ij}^\perp (\nabla_l \Phi_{il}^m) (\nabla_m \Phi_{jm}^m) + \frac{1}{2} \tau \delta_{ij}^\perp W_i W_j \end{aligned}$$

$$\begin{aligned} & + \zeta_{ij}^n A_{ij} \nabla_k \lambda_k^s + \xi^m \delta_{ij}^\perp W_i \nabla_l \Phi_{jl}^m \\ & + \xi_{ijk}^D W_k \Psi_{ij}. \end{aligned} \quad (\text{A1})$$

There are two new material tensors related to transient elasticity. First, ξ_{ijkl} , describing strain relaxation, has the form of the elasticity tensor, Eq. (22), with five relaxation coefficients. Second, the cross coupling to relative rotations is given by ξ_{ijk}^D , where

$$\xi_{ijk}^D = \xi^D (m_i^0 \delta_{jk}^\perp + m_j^0 \delta_{ik}^\perp), \quad (\text{A2})$$

which is symmetric in i and j .

All other terms and material tensors have already been discussed in Sec. III D.

The currents and quasicurrents read

$$j_i^{\sigma D} = -\kappa_{ij} \nabla_j T, \quad (\text{A3})$$

$$\sigma_{ij}^D = -v_{ijkl} A_{kl} - \zeta_{ij}^n \nabla_k \lambda_k^s, \quad (\text{A4})$$

$$Y_{ij}^D = \xi_{ijkl} \Psi_{kl} + \xi_{ijk}^D W_k, \quad (\text{A5})$$

$$I_\varphi^D = -\zeta \nabla_k \lambda_k^s - \zeta_{ij}^n A_{ij}, \quad (\text{A6})$$

$$Z_i^D = \tau \delta_{ij}^\perp W_j + \xi^m \delta_{ij}^\perp \nabla_l \Phi_{jl}^m + \xi_{kli}^D \Psi_{kl}, \quad (\text{A7})$$

$$X_i^D = b \delta_{ij}^\perp \nabla_k \Phi_{jk}^m + \xi^m \delta_{ij}^\perp W_j. \quad (\text{A8})$$

2. Distorted A phase

Due to the existence of a preferred direction that is odd under time reversal, there are many contributions in the reversible currents and quasicurrents:

$$j_i^{\sigma R} = -\kappa_{ij}^R \nabla_j T, \quad (\text{A9})$$

$$\begin{aligned} \sigma_{ij}^R = & -\Psi_{ij} - \frac{1}{2} \lambda_{kji} \nabla_l \Phi_{kl}^l - \varphi_{ijkl}^R A_{kl} \\ & + \xi_{ij}^{\sigma R} W - \lambda_{ij}^L \nabla_k \lambda_k^s, \end{aligned} \quad (\text{A10})$$

$$Y_{ij}^R = -A_{ij} + \xi_{ij}^{YR} W - \bar{\varphi}_{ijkl}^R \Psi_{kl} - \zeta_{kij}^{XR} \nabla_l \Phi_{kl}^l,$$

$$X_i^R = -\beta_{ij}^R \nabla_k \Phi_{jk}^l - \lambda_{ijk} A_{kj} + \xi_i^{XR} W - \zeta_{ijk}^{XR} \Psi_{jk}, \quad (\text{A11})$$

$$I_\varphi^R = \mu - \lambda_{ij}^L A_{ij}, \quad (\text{A12})$$

$$Z^R = -\xi_j^{XR} \nabla_l \Phi_{jl}^l - \xi_{ij}^{\sigma R} A_{ij} - \xi_{ij}^{YR} \Psi_{ij}, \quad (\text{A13})$$

There is a new reversible self-coupling term characterized by the antisymmetric material tensor $\bar{\varphi}_{ijkl}^R = -\bar{\varphi}_{kl ij}^R$. It has the same form as φ_{ijkl}^R in Eq. (85). Other cross-couplings involving elastic stresses are described by ζ_{ijk}^{XR} and ξ_{ij}^{YR} , which are of the form of Eqs. (68) and (86), respectively.

All other terms and material tensors have already been discussed in Sec. IV B.

For the dissipative dynamics we find

$$\begin{aligned} R = & \frac{1}{2} \kappa_{ij} (\nabla_i T) (\nabla_j T) + \frac{1}{2} v_{ijkl} A_{ij} A_{kl} \\ & + \frac{1}{2} \xi_{ijkl} \Psi_{ij} \Psi_{kl} \\ & + \frac{1}{2} \zeta (\nabla_i \lambda_i^s) (\nabla_j \lambda_j^s) + \zeta_{ij}^n (\nabla_k \lambda_k^s) A_{ij} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} b_{ij} (\nabla_l \Phi_{il}^l) (\nabla_m \Phi_{jm}^l) + \xi^m \zeta_i^0 W (\nabla_l \Phi_{il}^l) \\
& + \frac{1}{2} \tau W^2 + \xi_{ij}^D W \Psi_{ij}, \tag{A14}
\end{aligned}$$

with the new dissipative material tensors ξ_{ijkl} and ξ_{ij}^D , which have the form of the viscosity tensor ν_{ijkl} and of Eq. (65), respectively.

All other terms and material tensors have already been discussed in Sec. IV C.

3. Distorted B phase

In this phase only the strain diffusion $R \sim (\nabla_k \Psi_{ik})(\nabla_l \Psi_{jl})$ is replaced by strain relaxation $R \sim \Psi_{ik} \Psi_{jl}$ leading to $Y_{ij} \sim \Psi_{kl}$.

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