

Macroscopic behavior of systems with an axial dynamic preferred direction

Helmut R. Brand^{1,2,a}, Harald Pleiner^{2,b} and D. Svenšek^{3,c}

¹ Theoretische Physik III, Universität Bayreuth, 95440 Bayreuth, Germany;

² Max-Planck-Institute for Polymer Research, POBox 3148, 55021 Mainz, Germany;

³ Department of Physics, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1111 Ljubljana, Slovenia

Received 4 August 2011 and Received in final form 19 October 2011

published online: 28 November 2011 – © EDP Sciences / Societa Italiana di Fisica / Springer-Verlag 2011

Abstract. We present the derivation of the macroscopic equations for systems with an axial dynamic preferred direction. In addition to the usual hydrodynamic variables we introduce the time derivative of the local preferred direction as a new variable and discuss its macroscopic consequences including new cross coupling terms. Such an approach is expected to be useful for a number of systems for which orientational degrees of freedom are important including, for example, the formation of dynamic macroscopic patterns shown by certain bacteria such as *Proteus mirabilis*. We point out similarities in symmetry between the additional macroscopic variable discussed here, and the magnetization density in magnetic systems as well as the so-called $\hat{\mathbf{l}}$ vector in superfluid $^3\text{He-A}$. Furthermore we investigate the coupling to a gel-like system for which one has the strain tensor and relative rotations between the new variable and the network as additional macroscopic variables.

1 Introduction

In active media – as they are dominant in biological systems – one finds, as a rule, dynamically a preferred direction, which disappears when the external driving force, typically a chemical reaction leading to concentration gradients, is vanishing. In the field of pattern formation in such media typical examples include bacterial growth as it has been studied for a number of systems including the bacterium *Proteus mirabilis* [1, 2]. Depending on the actual system under consideration these dynamic preferred directions can be polar or axial in nature. For example, for the motion of motors along actin filaments one can distinguish between head and tail thus giving rise to a polar preferred direction dynamically [3, 4]. If one considers the motion of flagella [5–9], which are accompanied by the rotation of one or, more typically, bundles of flagella, one has an axial dynamic preferred direction closely linked to a broken orientational symmetry due to a chemical reaction driving this rotation.

From a macroscopic point of view there are significant differences between equilibrium systems with broken translational symmetries as they exist in all crystals and broken rotational symmetries as they are the rule for all liquid crystalline systems [10]. In the case of crystals one has the density of linear momentum as the generator of the broken symmetries while for broken rotational symmetries the angular momentum, an axial vector, serves as the generator [11]. From systems close to local thermodynamic equilibrium it is well known that the two types

of hydrodynamic systems have qualitatively different behavior with respect to their dynamic equations, their mode structure, their nonlinear behavior and instabilities [10–13]. Equilibrium systems with broken rotational symmetry can have a polar or an axial preferred direction. The former include, for example, ferroelectric and antiferroelectric crystals and liquid crystals, while equilibrium preferred axial directions can be found in ferromagnetic systems. The nematic director \mathbf{n} behaves in many respects similar to an axial vector, although it is not a vector due to the general \mathbf{n} to $-\mathbf{n}$ invariance.

It therefore appears to be natural to investigate for active (driven) systems the two analogous cases of axial and polar preferred direction. In this paper it is our goal to investigate as a first step the macroscopic behavior of a system which has an axial preferred direction dynamically and thus is closely linked to active systems showing rotational motions as an essential ingredient of their dynamics.

It is important to emphasize that the work presented here is rather different from the traditional work on liquid crystals, nematics, cholesterics and smectics in the sense that the orientational preferred direction exists only dynamically, but vanishes as soon as the thermodynamic driving force vanishes [12, 13]. Thus the systems we consider have no preferred direction in equilibrium in contrast to all conventional liquid crystalline phases. However, we will not describe the process of switching on or off the driving force, but will assume that the driving force is always on, i.e. the system is in its active state showing the preferred dynamic direction.

As it turns out fluid and gel-like active systems show in their macroscopic behavior a number of similarities with other complex fluids analyzed before. One example is superfluid $^3\text{He-A}$, which has an axial preferred direction in real space

^a e-mail: brand@uni-bayreuth.de

^b e-mail: pleiner@mpip-mainz.mpg.de

^c e-mail: daniel.svenssek@fmf.uni-lj.si

breaking time reversal symmetry - in addition to a preferred direction in spin space [14, 15]. A second more recent example are uniaxial magnetic gels [16, 17], which have in equilibrium a nonvanishing value of the magnetization density as a macroscopic quantity that breaks time reversal invariance. In addition, one can have in the latter system relative rotations between the two ‘subsystems’, the magnetic fluid component and the gel network, as macroscopic variables [18].

A typical example relevant to the present paper are flagella found in many biological systems, which are already complex objects by themselves, but can nevertheless form bundles which reveal strongly collective behavior. The latter property shows that what is of interest here also forms a bridge between mesoscopic and macroscopic behavior. We would like to stress, however, that we will not address any issue at a microscopic level and/or single molecule behaviour, but that the focus here is completely on collective behaviour and sufficiently long time scales and sufficiently large spatial scales.

We mention that there is also a body of literature using modified nematodynamic equations to describe active systems including nonpolar [19] as well as polar [20] aspects.

The present paper is organized as follows. In section II we describe in detail the choice of the additional macroscopic variables and the underlying motivation. In section III we derive the resulting macroscopic equations followed in section IV by a discussion of the modifications resulting from the presence of a gel. In section V we investigate some simple solutions of the macroscopic equations presented and in section VI we examine the influence of a macroscopic collective handedness, which is clearly of high interest for biological systems. Finally we present in section VII a brief summary and conclusions.

2 Hydrodynamic and macroscopic variables

2.1 Hydrodynamic variables

As truly conserved hydrodynamic quantities we have in the type of system we consider here the mass density, ρ , the energy density, ε , and the momentum density, \mathbf{g} . Since most of the systems of interest have several (say n) components, there are n conserved densities as truly hydrodynamic quantities for mixtures without chemical reactions. We will describe them in the following by the total density ρ and $n - 1$ concentrations ϕ_α . When chemical reactions are taken into account, the concentrations ϕ_α must be transferred from the list of truly hydrodynamic variables to the class of macroscopic variables, which relax on a sufficiently long, but finite time scale. In section IV we will also consider the coupling terms arising for networks or gels. In this case we also keep the strain field, ε_{ij} , in our list of hydrodynamic (chemically permanently cross-linked network) and/or macroscopic (when we make allowance for the existence of a transient network) variables.

2.2 A new macroscopic variable

In this paper we are interested in a type of behavior not considered before in macroscopic descriptions, namely in systems which have an axial preferred direction only dynamically. That

is we consider systems in nonequilibrium situations for which there is no preferred direction in true equilibrium.

A typical example that comes to mind are the patterns formed in bacterial growth, for example for the bacterium *Proteus mirabilis* [2]. As an initial condition the gel is prepared in a Petri dish containing a certain food concentration homogeneously across the whole experimental cell. Then the gel at the center of the Petri dish is inoculated with the bacterial suspension. As long as there is food and the other experimental conditions - such as the stiffness of the agar in the Petri dish - are favorable, the bacteria elongate, start to move and form macroscopic patterns over extended periods of time of the order of one day. Once the food has disappeared, the elongated bacteria return to their original shape and random orientations and the macroscopic patterns disappear as well. Both, the onset of pattern formation in this system as well as the disappearance of the patterns has been modeled recently [21].

In many biological systems one detects preferred directions only dynamically on many length and time scales, while there is no static preferred direction. On large length scales one observes, for example, bird flocks and fish swarms [22–25]. There is one straightforward, but important feature: before the birds start their flight, that is as long as they are on the ground, there is typically no preferred direction at all. Only after they start to fly and form a flock there is a dynamic preferred direction. After the birds land and disperse on the ground, there is again no preferred direction.

The same applies to the patterns formed by the bacterium *Proteus mirabilis* discussed above: they form target waves and spiral patterns only as long transients on intermediate length (\sim cm) and time (\sim hours) scales.

Going to smaller spatial scales, the beating of cilia and flagella is observed in many systems dynamically on a length scale of micro meters (μm) [26–28]. For the latter class of systems one has on average an angular velocity from the dynamics of the cilia and flagella, while for the other systems mentioned above a mean velocity (the case of bird flocks and fish swarms) or a velocity difference with respect to the background (the case of pattern formation in *Proteus mirabilis*) results.

Here we focus on the description of systems, which are characterized by a mean angular velocity including the angular motion of cilia and flagella. This quantity, which we will call \mathbf{W} in the following, transforms like an axial vector. It is even under parity, odd under time reversal and invariant under Galilei transformations. For the case of a mean velocity, as it applies in the other systems sketched out above, on the contrary, the state variable transforms like a velocity. It is a polar vector, which is odd under parity and time reversal and transforms like a velocity under Galilei transformations. The macroscopic behavior of such a system will be investigated in a separate paper [29].

The vector \mathbf{W} has in general three independent components in three dimensions (3 D). Drawing on our experience with superfluid $^3\text{He-A}$ [14, 15] we can anticipate immediately that the behavior under parity and Galilei transformations will lead to important consequences in the macroscopic dynamic behavior: \mathbf{W} behaves under parity, time reversal and Galilei transformations like the $\hat{\mathbf{I}}$ vector of $^3\text{He-A}$, while an average velocity has under these transformations the same behavior as,

for example, the superfluid velocity, \mathbf{v}_s in ${}^3\text{He} - A$. We also point out that under the transformations just discussed, \mathbf{W} has the same properties as the magnetization density, \mathbf{M} , in a magnetic system, which is also an axial vector.

As for many other systems, including polar nematic liquid crystals and magnetic systems, one can decompose \mathbf{W} into a magnitude, $|\mathbf{W}| \equiv W$, and into its direction, $w_i = W_i/W$. The former can be interpreted as the degree of the dynamic ordering. The direction w_i , which exists only dynamically, will be used throughout to construct property tensors reflecting the locally uniaxial properties of our systems via the use of w_i itself and the transverse Kronecker delta, δ_{ij}^\perp , with $\delta_{ij}^\perp = \delta_{ij} - w_i w_j$, which projects onto the plane perpendicular to w_i . We have checked that such a description leads to the same results for macroscopic dynamics as using \mathbf{W} . Only when the vicinity of defect cores is considered, the use of \mathbf{W} is more advantageous avoiding any singularities.

Thus in the fluid systems we consider we have \mathbf{W} as an additional macroscopic variable. Just as for magnetic systems with the magnetization density, \mathbf{M} , and for superfluid ${}^3\text{He} - A$ with $\hat{\mathbf{l}}$, there is no \mathbf{W} to $-\mathbf{W}$ symmetry here.

We note that the approach presented here must be contrasted to the case of the macroscopic description of usual nematic liquid crystals [12], where one has an equilibrium preferred direction, usually denoted as director, \mathbf{n}^0 , and also to that of polar nematics [30–32], where one has in equilibrium a polar preferred direction, \mathbf{p}^0 . In both cases, the macroscopic variables are the deviations from the preferred equilibrium direction, $\delta\mathbf{n} = \mathbf{n} - \mathbf{n}^0$ and $\delta\mathbf{p} = \mathbf{p} - \mathbf{p}^0$, respectively.

3 Derivation of macroscopic equations

3.1 Statics and thermodynamics

To get the static properties of the fluid systems under consideration we use the local first law of thermodynamics relating changes in the entropy density σ to changes in the hydrodynamic variables and the new macroscopic variable, W_i , discussed above. We have for the Gibbs relation

$$d\varepsilon = Td\sigma + \mu d\rho + \mu_\alpha d\phi_\alpha + v_i dg_i + h_i'^W dW_i + \Phi_{ij}^W d(\nabla_j W_i) \quad (1)$$

where α counts the different concentrations running from 1 to $n - 1$ and where summation over such Greek indices occurring twice is always assumed. In addition, the Einstein summation convention for Cartesian components is always assumed. In Eq.(1) the thermodynamic quantities, temperature T , chemical potential μ , relative chemical potentials μ_α , velocity v_i , the molecular fields $h_i'^W$ and Φ_{ij}^W , are defined as partial derivatives of the energy density with respect to the appropriate variables [13]. If we neglect surface effects and integrate Eq.(1) by parts we can obtain an expression for the Gibbs relation that we want to use throughout the rest of this paper

$$d\varepsilon = Td\sigma + \mu d\rho + \mu_\alpha d\phi_\alpha + v_i dg_i + h_i'^W dW_i \quad (2)$$

where the molecular field $h_i'^W$ is given by $h_i'^W = h_i'^W - \nabla_j \Phi_{ij}^W$.

To determine the thermodynamic conjugate variables we need an expression for the local energy density. This energy

density must be invariant under time reversal as well as under parity and it must be invariant under rigid rotations, rigid translations and covariant under Galilei transformations. The expression for the energy density is assumed to be convex for stability reasons. Taking into account these symmetry arguments we write down an expansion for the generalized energy density up to second order in the variables that describe deviations from its local minimum and where we have also incorporated some interesting cubic coupling terms

$$\begin{aligned} \varepsilon = & \frac{\alpha}{2}(W_i - W_0 w_i)^2 + \frac{1}{2} K_{ijkl} (\nabla_i W_j) (\nabla_k W_l) \\ & + \sigma_{ijk}^\sigma (\nabla_i W_j) (\nabla_k \delta\sigma) + \sigma_{ijk}^\rho (\nabla_i W_j) (\nabla_k \delta\rho) \\ & + \sigma_{ijk}^\alpha (\nabla_i W_j) (\nabla_k \delta\phi_\alpha) \\ & + c_{\rho\rho} (\delta\rho)^2 + c_{\sigma\sigma} (\delta\sigma)^2 + c_{\alpha\beta} (\delta\phi_\alpha) (\delta\phi_\beta) \\ & + c_{\rho\alpha} (\delta\rho) (\delta\phi_\alpha) + c_{\rho\sigma} (\delta\rho) (\delta\sigma) + c_{\sigma\alpha} (\delta\sigma) (\delta\phi_\alpha) \\ & + \frac{1}{2\rho} g_i g_i + c_{ijk} g_i (\nabla_j W_k) \\ & + (a_\alpha \delta\phi_\alpha + a_\sigma \delta\sigma + a_\rho \delta\rho) (W_i - W_0 w_i)^2 \end{aligned} \quad (3)$$

Apart from the energy density of a normal fluid binary mixture, Eq.(4) contains the energy density of normal fluid mixtures with $(n - 1)$ concentrations, ϕ_α for a n -component mixture. There is a stiffness coefficient α for \mathbf{W} and a Frank type of gradient free energy for the rotations of \mathbf{W} . The former describes the energetic penalty, when the degree of dynamic ordering is different from W_0 , the one induced by the driving force that puts the system into its active state. We would like to stress that because of the negative time reversal property of W_i – a coupling between the curl of \mathbf{W} and the momentum density mediated by the tensor c_{ijk} can arise, which takes the form $c_{ijk} = (c_{||} w_i w_l + c_\perp \delta_{il}^\perp) \epsilon_{ljk}$. This kind of coupling is very similar to one of the couplings appearing in superfluid ${}^3\text{He} - A$ first introduced by R. Graham [14]. In this system one defines an axial vector \mathbf{l} parallel to the direction of the net orbital momentum of the helium pairs. This vector does not commute with the total angular momentum vector and therefore this variable breaks the continuous rotational symmetry spontaneously similar to the variable W_i in our system. The source free part of the momentum density of ${}^3\text{He} - A$ is proportional to the curl of this vector \mathbf{l} while the proportionality is given by $\frac{\hbar}{2m}$ and a coupling tensor c_{ij} .

Since the system we discuss is uniaxial, all property tensors will therefore take a uniaxial form.

The tensor K_{ijkl} describes contributions to the local energy density due to spatial changes of the direction of the vector W_i as well as of its magnitude. We get six independent constants for this coupling

$$\begin{aligned} K_{ijkl} = & \frac{1}{2} K_1 (\delta_{ij}^\perp \delta_{kl}^\perp + \delta_{il}^\perp \delta_{jk}^\perp) + K_2 w_p \epsilon_{pij} w_q \epsilon_{qkl} \\ & + K_3 w_k w_i \delta_{lj}^\perp \\ & + K_4 w_i w_j w_k w_l + K_5 w_j w_l \delta_{ik}^\perp \\ & + \frac{1}{4} K_6 (w_i w_l \delta_{kj}^\perp + w_j w_k \delta_{il}^\perp + w_i w_j \delta_{kl}^\perp + w_k w_l \delta_{ij}^\perp) \end{aligned} \quad (4)$$

For the couplings between the gradients of the scalars ρ , c_i and σ and the gradients of the vector W_i , the tensors take the

following form

$$\sigma_{ijk}^f = \sigma_1^f w_i w_j w_k + \sigma_2^f w_j \delta_{ik}^\perp + \sigma_3^f (w_i \delta_{jk}^\perp + w_k \delta_{ij}^\perp) \quad (5)$$

where f can be either ρ, σ or α .

We now give the expressions for the conjugated variables in terms of the hydrodynamic and macroscopic variables. They are defined as partial derivatives with respect to the appropriate variable, while all the other variables are kept constant, denoted by dots at the brackets in the following. We obtain

$$v_i = \left(\frac{\partial \varepsilon}{\partial g_i} \right)_{\dots} = \frac{1}{\rho} g_i + c_{ijk} \nabla_j W_k \quad (6)$$

$$\begin{aligned} h_i^W &= \left(\frac{\partial \varepsilon}{\partial W_i} \right)_{\dots} \\ &= (\alpha + 2a_\alpha \delta \phi_\alpha + 2a_\sigma \delta \sigma + 2a_\rho \delta \rho) \\ &\quad \times (W_i - W_0 w_i) \end{aligned} \quad (7)$$

$$\begin{aligned} \Phi_{ij}^W &= \left(\frac{\partial \varepsilon}{\partial (\nabla_j W_i)} \right)_{\dots} \\ &= K_{ijkl} \nabla_k W_l + \sigma_{ijk}^\rho (\nabla_k \delta \rho) + \sigma_{ijk}^\sigma (\nabla_k \delta \sigma) \\ &\quad + \sigma_{ijk}^\alpha (\nabla_k \delta \phi_\alpha) + c_{ijk} g_k \end{aligned} \quad (8)$$

$$\begin{aligned} \delta T &= \left(\frac{\partial \varepsilon}{\partial \delta \sigma} \right)_{\dots} \\ &= 2c_{\sigma\sigma} \delta \sigma + c_{\rho\sigma} \delta \rho + c_{\sigma\alpha} \delta \phi_\alpha \\ &\quad + a_\sigma (W_i - W_0 w_i)^2 \end{aligned} \quad (9)$$

$$\begin{aligned} \delta \mu &= \left(\frac{\partial \varepsilon}{\partial \delta \rho} \right)_{\dots} \\ &= 2c_{\rho\rho} \delta \rho + c_{\rho\alpha} \delta \phi_\alpha + c_{\rho\sigma} \delta \sigma \\ &\quad + a_\rho (W_i - W_0 w_i)^2 \end{aligned} \quad (10)$$

$$\begin{aligned} \delta \mu_\alpha &= \left(\frac{\partial \varepsilon}{\partial \delta c_i} \right)_{\dots} \\ &= 2c_{\alpha\beta} \delta \phi_\beta + c_{\alpha\rho} \delta \rho + c_{\alpha\sigma} \delta \sigma \\ &\quad + a_\alpha (W_i - W_0 w_i)^2 \end{aligned} \quad (11)$$

We used integration by parts to obtain expression (2) for the local energy density, where the new molecular field h_i^W was given by $h_i^W = h_i^{\prime W} - \nabla_j \Phi_{ij}^W$. If we use Eqs. (7,8) we find

$$\begin{aligned} h_i^W &= (\alpha + 2a_\alpha \delta \phi_\alpha + 2a_\sigma \delta \sigma + 2a_\rho \delta \rho) (W_i - W_0 w_i) \\ &\quad - (\nabla_j K_{ijkl}) (\nabla_k W_l) - K_{ijkl} (\nabla_j \nabla_k W_l) \\ &\quad - (\nabla_j \sigma_{ijk}^\rho) (\nabla_k \delta \rho) - \sigma_{ijk}^\rho (\nabla_j \nabla_k \delta \rho) \\ &\quad - (\nabla_j \sigma_{ijk}^\sigma) (\nabla_k \delta \sigma) - \sigma_{ijk}^\sigma (\nabla_j \nabla_k \delta \sigma) \\ &\quad - (\nabla_j \sigma_{ijk}^\alpha) (\nabla_k \delta \phi_\alpha) - \sigma_{ijk}^\alpha (\nabla_j \nabla_k \delta \phi_\alpha) \\ &\quad - (\nabla_j c_{ijk}) g_k - c_{ijk} (\nabla_j g_k) \end{aligned} \quad (12)$$

3.2 Dynamic equations

To determine the dynamics of the variables we take into account that the first class of our set of variables contains conserved quantities that obey a local conservation law while the dynamics of the variable W_i can be described by a simple balance equation where the counter term to the temporal change of

the quantity is called a quasicurrent [13]. As a set of dynamical equations we get

$$\partial_t \rho + \nabla_i g_i = 0 \quad (13)$$

$$\partial_t \sigma + \nabla_i (\sigma v_i) + \nabla_i j_i^\sigma = \frac{R}{T} \quad (14)$$

$$\rho \partial_t \phi_\alpha + \rho v_i \nabla_i \phi_\alpha + \nabla_i j_i^\alpha = 0 \quad (15)$$

$$\partial_i g_i + \nabla_j (v_j g_i + \delta_{ij} p + \sigma_{ij}^{th} + \sigma_{ij}) = 0 \quad (16)$$

$$\partial_t W_i + v_j \nabla_j W_i + \frac{1}{2} (\mathbf{W} \times [\nabla \times \mathbf{v}])_i + X_i = 0 \quad (17)$$

where we introduced

$$\sigma_{ij}^{th} = +\frac{1}{2} (\Psi_{jk} \epsilon_{ki} + \Psi_{ik} \epsilon_{kj}) \quad (18)$$

In Eq.(18) we implemented the requirement that the energy density should be invariant under rigid rotations [13].

The pressure p in Eq.(16) is given by $\partial E / \partial V$ and reads for our system

$$p = -\varepsilon + \mu\rho + T\sigma + \mathbf{v} \cdot \mathbf{g} \quad (19)$$

In the equation for the entropy density (14) we introduced R , the dissipation function which represents the entropy production of the system. Due to the second law of thermodynamics R must satisfy $R \geq 0$. For reversible processes this dissipation function is equal to zero while for irreversible processes it must be positive. In the following we will split the currents and quasicurrents into reversible parts (denoted with a superscript R) and irreversible parts (superscript D).

3.3 Reversible dynamics

If we again make use of the symmetry arguments mentioned above and use Onsager's relations we obtain the following expressions for the reversible currents up to linear order in the thermodynamic forces

$$g_i = \rho v_i - \rho c_{ijk} \nabla_j W_k \quad (20)$$

$$j_i^{\sigma R} = -\kappa_{ij}^R \nabla_j T - D_{ij}^{\alpha TR} \nabla_j \mu_\alpha \quad (21)$$

$$j_i^{\alpha R} = -D_{ij}^{\alpha\beta R} \nabla_j \mu_\beta - D_{ij}^{\alpha TR} \nabla_j T \quad (22)$$

$$\sigma_{ij}^R = -c_{ijk}^R h_k^W - \nu_{ijkl}^R A_{kl} \quad (23)$$

$$X_i^R = b_{ij}^R h_j^W - c_{jki}^{RJ} A_{jk} \quad (24)$$

The additional term in the momentum density already appeared in superfluid ${}^3\text{He} - A$ and we will discuss one of the consequences of this coupling later. The material tensors in Eqs. (21) and (22) all have to be odd under time reversal, because the currents have to be odd under time reversal. They are of the form

$$F_{ij} = F \epsilon_{ijk} w_k \quad (25)$$

where F is either κ^R , $D^{\alpha TR}$, or $D^{\alpha\beta R}$. The antisymmetric form $F_{ij} = -F_{ji}$ guarantees a vanishing entropy production required for reversible currents, if in addition $D^{\alpha\beta R} = D^{\beta\alpha R}$

is assumed. Furthermore we find for the coupling terms in the stress tensor

$$c_{ijk}^{RJ} = c_1^{RJ} (w_i \delta_{jk}^\perp + w_j \delta_{ik}^\perp) + c_2^{RJ} w_k \delta_{ij}^\perp + c_3^{RJ} w_i w_j w_k \quad (26)$$

$$\nu_{ijkl}^R = \nu_1^R (\epsilon_{ikp} \delta_{jl}^\perp + \epsilon_{jlp} \delta_{ik}^\perp + \epsilon_{ilp} \delta_{jk}^\perp + \epsilon_{jkp} \delta_{il}^\perp) w_p + \nu_2^R (\epsilon_{ikp} w_j w_l + \epsilon_{jlp} w_i w_k + \epsilon_{ilp} w_j w_k + \epsilon_{jkp} w_i w_l) w_p \quad (27)$$

We are now left with the tensor coupling the molecular field h_i^W to the quasicurrent for W_i , which takes the form

$$b_{ij}^R = b^R \epsilon_{ijk} w_k \quad (28)$$

As a consequence of the existence of a dynamical preferred direction, there is a contribution in the reversible stress tensor, Eq.(23),

$$\sigma_{ij}^R = \alpha W_0 (c_2^{RJ} \delta_{ij}^\perp + c_3^{RJ} w_i w_j) \quad (29)$$

that resembles the active term in the nematodynamic description of active systems [19].

3.4 Irreversible dynamics and entropy production

We can use the dissipation function R to derive the irreversible currents and quasicurrents. Thereby, R/T is the amount of entropy produced within a unit volume per unit time. One can expand the function R into the thermodynamic forces using the same symmetry arguments as in the case of the energy density. We obtain

$$R = \frac{1}{2} \kappa_{ij} (\nabla_i T) (\nabla_j T) + D_{ij}^{\alpha T} (\nabla_i T) (\nabla_j \mu_\alpha) + \frac{1}{2} D_{ij}^{\alpha\beta} (\nabla_i \mu_\alpha) (\nabla_j \mu_\beta) + \frac{1}{2} \nu_{ijkl} A_{ij} A_{kl} + c_{ijk}^J A_{ij} h_k^W + \frac{1}{2} b_{ij} h_i^W h_j^W \quad (30)$$

where we have introduced the transport tensors. The tensors κ_{ij} and $D_{ij}^{\alpha T}$, $D_{ij}^{\alpha\beta}$, and b_{ij} take the uniaxial symmetric form

$$G_{ij} = G_{\parallel} w_i w_j + G_{\perp} \delta_{ij}^\perp \quad (31)$$

and, additionally, there is $D_{ij}^{\alpha\beta} = D_{ij}^{\beta\alpha}$. For the viscosity tensor ν_{ijkl} we obtain

$$\nu_{ijkl} = \nu_1 \delta_{ij}^\perp \delta_{kl}^\perp + \nu_2 (\delta_{ik}^\perp \delta_{jl}^\perp - \frac{1}{2} \delta_{ij}^\perp \delta_{kl}^\perp + \delta_{il}^\perp \delta_{jk}^\perp - \frac{1}{2} \delta_{ij}^\perp \delta_{kl}^\perp) + \nu_3 w_i w_j w_k w_l + \nu_4 (w_i w_j \delta_{kl}^\perp + w_k w_l \delta_{ij}^\perp) + \nu_5 (w_i w_k \delta_{jl}^\perp + w_i w_l \delta_{jk}^\perp + w_j w_k \delta_{il}^\perp + w_j w_l \delta_{ik}^\perp) \quad (32)$$

Finally, the tensor c_{ijk}^J takes the form

$$c_{ijk}^J = c^J (w_i \epsilon_{jkl} + w_j \epsilon_{ikl}) w_l \quad (33)$$

To obtain the dissipative parts of the currents and quasicurrents we take the partial derivative with respect to the appropriate

thermodynamic force

$$j_i^{\sigma D} = - \left(\frac{\partial R}{\partial (\nabla_i T)} \right) \dots = -\kappa_{ij} (\nabla_j T) - D_{ij}^{\alpha T} (\nabla_j \mu_\alpha) \quad (34)$$

$$j_i^{\alpha D} = - \left(\frac{\partial R}{\partial (\nabla_j \mu_\alpha)} \right) \dots = -D_{ij}^{\alpha\beta} (\nabla_j \mu_\beta) - D_{ij}^{\alpha T} (\nabla_j T) \quad (35)$$

$$\sigma_{ij}^D = - \left(\frac{\partial R}{\partial (\nabla_j v_i)} \right) \dots = -\nu_{ijkl} A_{kl} - c_{ijk}^J h_k^W \quad (36)$$

$$X_i^D = \left(\frac{\partial R}{\partial h_i^W} \right) \dots = b_{ij} h_j^W + c_{kli}^J A_{kl} \quad (37)$$

4 Macroscopic equations in the presence of a gel

In this section we analyze how the analysis given so far must be modified to incorporate the hydrodynamic and macroscopic variables associated with the presence of a gel or an elastomer. As already discussed in Sec. 2 we have as additional variable the strain field, ϵ_{ij} . Inspired by the macroscopic physical properties of nematic elastomers [33], uniaxial magnetic gels [18] and nematic liquid crystals with D_{2d} symmetry [34], one can introduce in close analogy also relative rotations, $\tilde{\Omega}_i$, in the present type of systems, if a gel is investigated. Relative rotations are introduced via

$$\tilde{\Omega}_i = \delta w_i - \Omega_i^\perp = \delta w_i - \frac{1}{2} w_j (\nabla_i u_j - \nabla_j u_i) \quad (38)$$

where the vector u_i describes the displacement field of the network. Since w_i is a unit vector, $w \cdot \delta w = 0$. The variable $\tilde{\Omega}_i$ is odd under time reversal and even under parity.

We thus find for the modified Gibbs relation

$$d\varepsilon = \dots + \Psi_{ij} d\epsilon_{ij} + \Sigma_i d\tilde{\Omega}_i \quad (39)$$

where in Eq.(39) the \dots indicate all the terms already given in Eq.(1). The same notation always applies in the following: \dots indicate all the terms present without the gel and given above. In Eq.(39) the additional thermodynamic quantities, the elastic stress Ψ_{ij} and the molecular field Σ_i , are defined as partial derivatives of the energy density with respect to the appropriate variables [13].

For the generalized energy density we have the additional terms

$$\varepsilon = \dots + \frac{1}{2} \mu_{ijkl} \epsilon_{ij} \epsilon_{kl} + \frac{1}{2} D_1 \tilde{\Omega}_i \tilde{\Omega}_i + D_2 (w_j \delta_{ik}^\perp + w_k \delta_{ij}^\perp) \tilde{\Omega}_i \epsilon_{jk} + \varepsilon_{ij} (\chi_{ij}^\sigma \delta \sigma + \chi_{ij}^\rho \delta \rho + \chi_{ij}^\alpha \delta \phi_\alpha) \quad (40)$$

The elastic tensor has five independent constants and takes the form

$$\mu_{ijkl} = \mu_1 \delta_{ij}^\perp \delta_{kl}^\perp + \mu_2 (\delta_{ik}^\perp \delta_{jl}^\perp - \frac{1}{2} \delta_{ij}^\perp \delta_{kl}^\perp + \delta_{il}^\perp \delta_{jk}^\perp - \frac{1}{2} \delta_{ij}^\perp \delta_{kl}^\perp) + \mu_3 w_i w_j w_k w_l + \mu_4 (w_i w_j \delta_{kl}^\perp + w_k w_l \delta_{ij}^\perp) + \mu_5 (w_i w_k \delta_{jl}^\perp + w_i w_l \delta_{jk}^\perp + w_j w_k \delta_{il}^\perp + w_j w_l \delta_{ik}^\perp) \quad (41)$$

There are two contributions to the energy density due to the coupling between the strain field and the new variables associated with the relative rotations. One is proportional to D_1 and the other proportional to D_2 . One can interpret these coefficients as a measure for the coupling strength of the dynamic preferred direction to the polymer network.

Now we are left with the couplings between the scalars ρ , c and σ and the strain field. These tensors take the form

$$\chi_{ij}^f = \chi_{||}^f w_i w_j + \chi_{\perp}^f \delta_{ij}^{\perp} \quad (42)$$

where f can be either ρ , σ or α .

We now give the modified expressions for the conjugated variables in terms of the hydrodynamic and macroscopic variables. We obtain

$$\begin{aligned} \Psi_{ij} = \left(\frac{\partial \varepsilon}{\partial \epsilon_{ij}} \right) \dots &= +\mu_{ijkl} \epsilon_{kl} + D_2 (w_j \delta_{ik}^{\perp} + w_i \delta_{kj}^{\perp}) \tilde{\Omega}_k \\ &+ \chi_{ij}^{\sigma} \delta \sigma + \chi_{ij}^{\rho} \delta \rho + \chi_{ij}^{\alpha} \delta \phi_{\alpha} \end{aligned} \quad (43)$$

$$\Sigma_i = \left(\frac{\partial \varepsilon}{\partial \tilde{\Omega}_i} \right) \dots = D_1 \tilde{\Omega}_i + D_2 (w_j \delta_{ik}^{\perp} + w_k \delta_{ij}^{\perp}) \epsilon_{jk} \quad (44)$$

$$\delta T = \left(\frac{\partial \varepsilon}{\partial \delta \sigma} \right) \dots = \dots + \chi_{ij}^{\sigma} \epsilon_{ij} \quad (45)$$

$$\delta \mu = \left(\frac{\partial \varepsilon}{\partial \delta \rho} \right) \dots = \dots + \chi_{ij}^{\rho} \epsilon_{ij} \quad (46)$$

$$\delta \mu_{\alpha} = \left(\frac{\partial \varepsilon}{\partial \delta \phi_{\alpha}} \right) \dots = \dots + \chi_{ij}^{\alpha} \epsilon_{ij} \quad (47)$$

and where the other thermodynamic forces are unchanged by the presence of a gel-like structure.

For the two additional macroscopic variables there are the following two additional dynamic equations

$$\partial_t \epsilon_{ij} + v_k \nabla_k \epsilon_{ij} + Y_{ij} = 0 \quad (48)$$

$$\partial_t \tilde{\Omega}_i + v_k \nabla_k \tilde{\Omega}_i + Z_i = 0 \quad (49)$$

For the reversible currents we obtain the following modified expressions up to linear order in the thermodynamic forces

$$j_i^{\sigma R} = \dots + \xi_{ij}^{TR} \nabla_l \Psi_{jl} \quad (50)$$

$$j_i^{\alpha R} = \dots + \xi_{ij}^{\alpha R} \nabla_l \Psi_{lj} \quad (51)$$

$$\sigma_{ij}^R = \dots - \Psi_{ij} + \xi_{ijk}^{\sigma R} \Sigma_k \quad (52)$$

$$\begin{aligned} Y_{ij}^R &= -A_{ij} + \xi_{ijk}^{YR} \Sigma_k \\ &+ \frac{1}{2} \lambda^W [\nabla_i (\nabla \times \mathbf{h}^W)_j + \nabla_j (\nabla \times \mathbf{h}^W)_i] \\ &- \frac{1}{2} [\nabla_i (\xi_{jkl}^R \nabla_l \Psi_{kl} + \xi_{jkl}^{TR} \nabla_k T + \xi_{jkl}^{\alpha R} \nabla_k \mu_{\alpha}) \\ &+ i \longleftrightarrow j] \end{aligned} \quad (53)$$

$$X_i^R = \dots + \lambda^W \epsilon_{ijk} \nabla_j \nabla_l \Psi_{kl} - c_{jki}^{RJ} A_{jk} + \xi_{ij}^{XR} \Sigma_j \quad (54)$$

$$Z_i^R = \tau_{ij}^R \Sigma_j - \xi_{ij}^{XR} h_j^W - \xi_{kli}^{\sigma R} A_{kl} - \xi_{kli}^{YR} \Psi_{kl} \quad (55)$$

We have now the additional current for the strains and the additional quasicurrent of relative rotations. The additional quasicurrent of relative rotations has counter terms in X_i^R , σ_{ij}^R and Y_{ij}^R . These terms describe the dynamic coupling of relative

rotations to the variable W_i , the momentum density and the network respectively. The first coupling – mediated by the tensor ξ_{ij}^{XR} – is a term that exists neither in nematic liquid crystalline elastomers [35] nor in superfluid $^3\text{He-A}$, while the second coupling – $\xi_{ijk}^{\sigma R}$ – already appeared in nematic liquid crystalline elastomers. The third coupling – ξ_{ijk}^{YR} – is also unknown from nematic elastomers and superfluid $^3\text{He-A}$. As the material tensors in Eqs. (21) and (22), the second rank reversible tensors have to be odd under time reversal and take the antisymmetric form

$$\begin{aligned} \xi_{ij}^{gR} &= \xi^{gR} \epsilon_{ijk} w_k, \quad \tau_{ij}^R = \tau^R \epsilon_{ijk} w_k, \\ \text{and} \quad \xi_{ij}^R &= \xi^R \epsilon_{ijk} w_k \end{aligned} \quad (56)$$

where g can either be α , X , or T . For the additional third rank reversible coupling tensors we find

$$\begin{aligned} \xi_{ijk}^{\sigma R} &= \xi^{\sigma R} (w_i \delta_{jk}^{\perp} + w_j \delta_{ik}^{\perp}), \\ \text{and} \quad \xi_{kli}^{YR} &= \xi^{YR} (w_k \epsilon_{lip} + w_l \epsilon_{kip}) w_p \end{aligned} \quad (57)$$

The dissipation function R acquires the terms

$$\begin{aligned} R &= \dots + \xi_{ij}^T (\nabla_i T) (\nabla_k \Psi_{jk}) + \xi_{ij}^{\alpha} (\nabla_i \mu_{\alpha}) (\nabla_k \Psi_{jk}) \\ &+ \xi_{ijk}^{\sigma} A_{ij} \Sigma_k + \frac{1}{2} \xi_{ij} (\nabla_k \Psi_{ik}) (\nabla_l \Psi_{jl}) \\ &+ \frac{1}{2} \tau_{ij} \Sigma_i \Sigma_j + \xi_{ij}^X \Sigma_i h_j^W \end{aligned} \quad (58)$$

where the additional tensors ξ_{ij}^T , ξ_{ij}^{α} and ξ_{ij} are of the uniaxial symmetric form, Eq.(42)

$$\xi_{ij} = \xi_{||} w_i w_j + \xi_{\perp} \delta_{ij}^{\perp} \quad (59)$$

while τ_{ij} and ξ_{ij}^X only have a transverse component

$$\tau_{ij} = \tau \delta_{ij}^{\perp} \quad \text{and} \quad \xi_{ij}^X = \xi^X \delta_{ij}^{\perp} \quad (60)$$

due to the fact that only the parts of the relative rotations perpendicular to the preferred direction contribute to the entropy production [35]. Finally the tensor ξ_{ijk}^{σ} takes the form

$$\xi_{ijk}^{\sigma} = \xi^{\sigma} (w_i \epsilon_{jkl} + w_j \epsilon_{ikl}) w_l \quad (61)$$

This leads to the modifications in the dissipative parts of the currents and quasicurrents

$$j_i^{\sigma D} = - \left(\frac{\partial R}{\partial (\nabla_i T)} \right)_{\dots} = \dots - \xi_{ij}^T \nabla_k \Psi_{jk} \quad (62)$$

$$j_i^{\alpha D} = - \left(\frac{\partial R}{\partial (\nabla_j \mu_\alpha)} \right)_{\dots} = \dots - \xi_{ij}^\alpha \nabla_k \Psi_{jk} \quad (63)$$

$$\sigma_{ij}^D = - \left(\frac{\partial R}{\partial (\nabla_j v_i)} \right)_{\dots} = \dots - \xi_{ijk}^\sigma \Sigma_k \quad (64)$$

$$Y_{ij}^D = \left(\frac{\partial R}{\partial \Psi_{ij}} \right)_{\dots} = -\frac{1}{2} \nabla_i (\xi_{jk} \nabla_l \Psi_{kl} + \xi_{jk}^T \nabla_k T + \xi_{jk}^\alpha \nabla_k \mu_\alpha) -\frac{1}{2} \nabla_j (\xi_{ik} \nabla_l \Psi_{kl} + \xi_{ik}^T \nabla_k T + \xi_{ik}^\alpha \nabla_k \mu_\alpha) \quad (65)$$

$$Z_i^D = \left(\frac{\partial R}{\partial \Sigma_i} \right)_{\dots} = \tau_{ij} \Sigma_j + \xi_{ij}^X h_j^W + \xi_{kli}^\sigma A_{kl} \quad (66)$$

$$X_i^D = \left(\frac{\partial R}{\partial h_i^W} \right)_{\dots} = \dots + \xi_{ij}^X \Sigma_j \quad (67)$$

5 The influence of a macroscopic hand

All biologically active systems are characterized by the presence of a microscopic hand. In this section we consider what happens if there is an intrinsic macroscopic handedness on average for a system with an axial dynamic preferred direction. This case is characterized by the presence of a pseudoscalar quantity, q_0 , which behaves like a scalar under all symmetry operations except for the ones containing parity. That is, the system breaks parity dynamically on the macroscopic scale. In the following we outline how the considerations given in the last sections have to be modified.

For the generalized energy density we find

$$\varepsilon = \varepsilon_0 + \tilde{K}_2 q_0 [\mathbf{w} \cdot (\nabla \times \mathbf{W}) + \frac{1}{2} q_0] - q_0 [\mathbf{w} \cdot (\nabla \times \mathbf{W}) + q_0] [\tau_\alpha \delta \phi_\alpha + \tau_\sigma \delta \sigma + \tau_\rho \delta \rho] \quad (68)$$

where ε_0 denotes all the nonchiral terms (cf. Eq.(3)) and where \tilde{K}_2 is usually taken as equal to the twist elastic constant K_2 [12], although there is no need for this special choice, which indeed leads to contradictions with experiments [36]. Just as for ordinary cholesterics, the spontaneous formation of a helical structure arises as a direct consequence of broken parity symmetry in a fluid system. The cross-coupling terms $\sim \tau_\alpha, \tau_\rho$ and τ_σ in the generalized energy (eq.(68)) are the analogues of the static Lehmann terms known from cholesteric materials [37–39]. This type of effects has become of central interest recently in connection to the Langmuir layer experiments at fluid - air interfaces performed by Tabe's group [40, 41], which are also thought to be of high relevance for transport phenomena associated with rotors on length scales microns (microscale).

Next we analyze the modifications in the dissipation function providing additional dissipative channels due to the presence of a macroscopic hand. We find

$$R = R_0 + q_0 \epsilon_{ijk} h_j w_k (\psi_\alpha \nabla_i \mu_\alpha + \psi_\sigma \nabla T + \psi_e E_i) + q_0 A_{ij} (\zeta_{ijk}^\sigma \nabla_k T + \zeta_{ijk}^e E_k + \zeta_{ijk}^\alpha \nabla_k \mu_\alpha) \quad (69)$$

where R_0 denotes all the nonchiral contributions given in Eq.(30) and where we have also incorporated the effect of an electric field, E_i . The contributions $\sim \psi_\alpha, \psi_\sigma$ and ψ_e are the analogues of the dissipative Lehmann terms familiar from cholesteric and chiral smectic liquid crystals [37–39]. The third rank tensors between symmetrized velocity gradients, A_{ij} , which are associated with extensional flow, and the thermodynamic forces $\nabla_k T$ (temperature gradients), E_k (electric fields) and $\nabla_k \mu_\alpha$ (gradients of the chemical potential associated with the concentration) take the form

$$\zeta_{ijk} = \zeta_1 \delta_{ij}^\perp w_k + \zeta_2 w_i w_j w_k + \zeta_3 (w_i \delta_{jk}^\perp + w_j \delta_{ik}^\perp) \quad (70)$$

An analogue of these terms has been written down in ref. [42] for the case of temperature gradients for cholesteric liquid crystals in an external magnetic field. We would like to emphasize, however, that in the present case these coupling terms arise due to the presence of the dynamic axial preferred direction and that therefore an additional external field is unnecessary. Naturally the additional coupling terms one can write down for an external magnetic field (replacing w by a magnetic field direction, \hat{H}), can be used to modify the strength as well as the direction of the type of cross-coupling effects just discussed.

To outline the consequences of one of these cross-coupling terms, we just consider the electric current, j_i^e

$$j_k^e = \frac{\delta R}{\delta E_k} = \zeta_{ijk}^e A_{ij} \quad (71)$$

and we read off immediately from eq.(71) that there is an electric current induced by flows. Or, to be more specific, for example, for a shear flow in the $x - y$ - plane ($\sim \zeta_\alpha^e$) we predict an electric current in the direction perpendicular to this plane.

To find out whether there are also reversible dynamic coupling terms due to broken parity in a system with an axial dynamic preferred direction, we investigate coupling terms between the group of macroscopic variables, which are even under time reversal ($\delta \sigma, \delta \phi_\alpha, P_i$) with the group of variables that are odd under time reversal (g_i, W_i). We obtain for the cross-coupling terms involving symmetrized velocity gradients and the stress tensor

$$j_i^{\sigma R} = \dots + q_0 \chi_{ijk}^\sigma A_{jk} \quad (72)$$

$$j_i^{eR} = \dots + q_0 \chi_{ijk}^e A_{jk} \quad (73)$$

$$j_i^{\alpha R} = \dots + q_0 \chi_{ijk}^\alpha A_{jk} \quad (74)$$

$$\sigma_{ij}^R = \dots + q_0 (\chi_{ijk}^\alpha \nabla_k \mu_\alpha + \chi_{ijk}^e E_k + \chi_{ijk}^\sigma \nabla_k T) \quad (75)$$

where \dots stands above as well as in the following for all the nonchiral terms given in previous sections and where we have for the structure of the third rank tensors

$$\chi_{ijk}^\xi = \chi^\xi (\epsilon_{ikm} w_j w_m + \epsilon_{jkm} w_i w_m) \quad (76)$$

From these equations we can conclude in a straightforward way, that, for example, a spatially varying vorticity leads to temperature variations. We also note that these terms also exist for ordinary cholesteric and chiral smectic liquid crystals,

since the χ_{ijk} contain an even number of w_i 's, so can be replaced by n_i 's in the case of cholesteric liquid crystals etc. Apparently this type of coupling terms has not been considered before, however.

For the coupling terms to the thermodynamic force associated with the dynamic axial preferred direction we find

$$j_i^{\sigma R} = \dots + q_0 \xi_{ij}^{\sigma} h_j \quad (77)$$

$$j_i^{eR} = \dots + q_0 \xi_{ij}^e h_j \quad (78)$$

$$j_i^{\alpha R} = \dots + q_0 \xi_{ij}^{\alpha} h_j \quad (79)$$

$$X_i^R = \dots + q_0 (\xi_{ij}^{\alpha} \nabla_j \mu_{\alpha} + \xi_{ij}^e E_j + \xi_{ij}^{\sigma} \nabla_j T) \quad (80)$$

with $\xi_{ij} = \xi_1 \delta_{ij}^{\perp} + \xi_2 w_i w_j$. There is no analogue of this type of cross-coupling terms in ordinary cholesteric liquid crystals due to the $n_i \rightarrow -n_i$ symmetry in the latter case.

We close this section by pointing out that the macroscopic system discussed here appears to be the first complex fluid, which breaks time reversal symmetry and has a macroscopic hand. Fluid systems showing macroscopic handedness include so far cholesteric and chiral smectic liquid crystalline phases, while as complex fluids breaking time reversal symmetry two of the superfluid phases of ^3He , namely $^3\text{He-A}$ and $^3\text{He-A}_1$ [43] come to mind.

6 Some simple solutions

In this section we discuss two possible experiments to detect some of the unusual cross-coupling effects present in systems with a macroscopic dynamic preferred direction of axial nature. In the gel system we find a reorientation of the preferred direction within the plane in which a strain rate is applied as well as out of this plane due to relative rotations and in a fluid system, which has additionally a macroscopic hand, we predict the rotation of the preferred direction in a temperature gradient.

6.1 Dynamic strain induced reorientation of the preferred direction

Here we discuss one effect closely related to the macroscopic variables associated with relative rotations. These variables describe, as already mentioned, the relative rotations between the preferred direction and the polymer network. We apply a constant strain rate and determine the change of orientation of the preferred direction. We take the dynamic preferred direction as the x -direction and the strain rate to be applied in the $x - y$ -plane.

We are looking for spatially homogeneous, stationary solutions. In this case the dynamic equations for the momentum density and the scalars ρ , σ and ϕ_{α} are satisfied automatically. The remaining dynamic equations (17), (48) and (49) require the quasi-currents X_i , Z_i and Y_{ij} to vanish, which leads to the conditions, cf. Eq.(24), (37), (53-55), (65-67)

$$X_i = b_{ij}^R h_j^W + \xi_{ij}^{XR} \Sigma_j - c_{jki}^{RJ} A_{jk} + b_{ij} h_j^W + \xi_{ij}^X \Sigma_j + c_{kli}^J A_{kl} = 0 \quad (81)$$

$$Z_i = \tau_{ij}^R \Sigma_j - \xi_{ij}^{XR} h_j^W - \xi_{kli}^{YR} \Psi_{kl} - \xi_{kli}^{\sigma R} A_{kl} + \tau_{ij} \Sigma_j + \xi_{ij}^X h_j^W + \xi_{kli}^{\sigma} A_{kl} = 0 \quad (82)$$

$$Y_{ij} = \xi_{ijk}^{YR} \Sigma_k - A_{ij} = 0 \quad (83)$$

where the homogeneity condition for the variables and conjugate quantities has been used.

We need to close the set of equations (81-83) by using the static constitutive relations, cf. Eqs.(12), (43) and (44), in the homogeneous case

$$h_j^W = \alpha \delta W_j \quad (84)$$

$$\Psi_{kl} = \mu_{klmn} \epsilon_{mn} + D_2 (w_k \delta_{lm}^{\perp} + w_l \delta_{km}^{\perp}) \tilde{\Omega}_m \quad (85)$$

$$\Sigma_k = D_1 \tilde{\Omega}_k + D_2 (w_m \delta_{kn}^{\perp} + w_n \delta_{km}^{\perp}) \epsilon_{mn} \quad (86)$$

We consider the case of a constant strain rate in the x -y plane, where $A_{kl} = S \delta_{ky} \delta_{lx}$. and where the magnitude of W_i is constant but only its direction is changing, which gives $\delta \mathbf{W} = W_0 \delta \mathbf{w}$. The reorientations $\delta \mathbf{w}$ as a function of the strain rate S are the sought after quantities. Performing a straightforward, but lengthy calculation, which closely parallels that given for the reorientation of the magnetization in a uniaxial magnetic gel [18], we find for the linear deviations from the orientation before the external force is applied

$$W_0 \delta w_y = - \frac{-b^R (\xi^X + 2\xi^{YR} c^J) + b_{\perp} (\xi^{XR} - 2\xi^{YR} c_1^{RJ})}{2\xi^{YR} \alpha (b_{\perp}^2 + b^{R2})} S \quad (87)$$

$$W_0 \delta w_z = - \frac{b_{\perp} (\xi^X + 2\xi^{YR} c^J) + b^R (\xi^{XR} - 2\xi^{YR} c_1^{RJ})}{2\xi^{YR} \alpha (b_{\perp}^2 + b^{R2})} S \quad (88)$$

while to first order (in S) the x component of \mathbf{w} is unchanged, since $w_x = \sqrt{1 - w_y^2 - w_z^2} = 1 - O(2)$.

For this set-up we thus predict a rotation of the dynamic preferred direction within the plane in which the strain rate is applied, as well as out of this plane. Both rotations are proportional to the applied external force (the rate S) and inversely proportional to the stiffness (α) of the preferred direction as can be seen from Eqs.(87,88). This effect is due to the variables associated with relative rotations, because all contributions are proportional to either ξ^{XR} , ξ^X or ξ^{YR} , which represent the dynamical coupling of relative rotations to the dynamic preferred direction and the strain field respectively. This effect only exists for an active state, where a finite W_0 is present and where the preferred direction is dynamical and of the axial type. The change of the dynamic preferred direction should be easily observable optically.

6.2 Rotation of the preferred direction in a temperature gradient

In the following we show that for a fluid system (without elastic and relative rotational degrees of freedom) with handedness and, thus, with a helical orientation of \mathbf{W} , a temperature gradient or a temperature difference applied along the helical axis

can generate a rotation of \mathbf{W} . To be specific we consider a thin film geometry as it is typically the case for dynamic investigations of freely suspended smectic films [44, 45] or for monolayers at air - liquid interfaces [40, 41]. We take the helical axis to be in z -direction and thus the film plane to be in the $x - y$ -plane. For simplicity we first assume the absence of in-plane flow and discuss its relevance later. For the in-plane preferred direction we then have $\hat{\mathbf{w}} = (\cos \varphi, \sin \varphi)$, and we assume a constant degree of ordering of the axial dynamic preferred direction \mathbf{W} .

Applying the temperature difference ΔT along the helical axis we find from Eqs. (68) - (80)

$$\dot{\varphi} = q_0 \Delta T (\tau_\sigma b_\perp + \psi_\sigma T_0 / C_v) \quad (89)$$

where T_0 is the mean temperature and where C_v is the specific heat at constant volume. τ_σ and ψ_σ are the static and dissipative Lehmann effects, respectively, associated with entropy variations. b_\perp is the analogue of the director diffusion constant γ_1^{-1} familiar from ordinary nematic and cholesteric liquid crystals. Eq.(89) describes a spatially homogeneous rotation of the preferred direction, for which the molecular field associated with the preferred direction, h_i^W vanishes. Thus we arrive at the conclusion that a close analogue of the Lehmann effect found first by Lehmann [46] is also possible for a phase, which breaks time reversal symmetry.

This prediction also allows a crucial test concerning the influence of a macroscopic hand. In the absence of macroscopic chirality the phenomenon predicted here does not exist, since the pseudoscalar, q_0 , vanishes identically and thus no uniform rotation of $\hat{\mathbf{w}}$ is possible in a uniform temperature difference or temperature gradient.

It is straightforward to check whether there are any changes in the analysis given above, when the assumption of the absence of flow is relaxed. Analyzing the influence of the dissipative contributions $\sim \zeta_{ijk}^\sigma$ from the dissipation function (eq.(69)) and of the reversible contributions $\sim \chi_{ijk}^\sigma$ in eq.(75), we find that there is no flow in the plane perpendicular to the helix for a homogeneous orientation of the dynamic preferred direction in this plane. Conversely, any deviation of the applied temperature gradient from the direction of the helical axis leads to a flow.

We close this subsection by pointing out, that an analogous effect is also expected if one replaces the temperature difference/gradient by a concentration difference/gradient, by an electric field or by a uniaxial pressure difference/gradient applied along the same direction.

7 Summary and conclusions

In this paper we have presented the macroscopic dynamic equations for an active medium with an axial dynamic preferred direction. This description is addressing systems for which one has collective rotational motions as, for example, for flagella. Our analysis shows that there are, from a hydrodynamic point of view, similarities to systems, which are physically quite different, namely to one of the superfluid phases of ^3He , $^3\text{He} - A$ and to uniaxial magnetic gels. To demonstrate the overlap with the latter system it turns out to be crucial to incorporate

the gel-like behavior leading to the strain field and to relative rotations as additional macroscopic variables. We also present the macroscopic equations for the case of an additional macroscopic hand leading to a combination of coupling terms in the hydrodynamic regime, which has never been found before for other macroscopic systems. As two examples we show how an applied strain rate or an external temperature gradient can give rise to an unconventional reorientation of the axial dynamic preferred direction.

H.R.B. and D.S. thank Mitsugu Matsushita for stimulating discussions and correspondence on pattern formation in *Proteus mirabilis*. D.S. acknowledges the support of the Slovenian Research Agency (ARRS Grant No. J1-0908). H.R.B. thanks the Deutsche Forschungsgemeinschaft for partial support of his work through the Forschergruppe FOR 608 ‘Nichtlineare Dynamik komplexer Kontinua’.

References

- [1] Y. Shimada, A. Nakahara, M. Matsushita, and T. Matsuyama, J. Phys. Soc. Jap. **64**, 1896 (1995).
- [2] K. Watanabe, J. Wakita, H. Itoh, H. Shimada, S. Kurosu, T. Ikeda, Y. Yamazaki, T. Matsuyama, and M. Matsushita, J. Phys. Soc. Jap. **71**, 650 (2002).
- [3] H. Honda, H. Nagashima, and S. Asakura, J. Mol. Biol. **191**, 131 (1986).
- [4] D. Humphrey, C. Duggan, D. Saha, D. Smith, and J. Kaes, Nature (London) **416**, 413 (2002).
- [5] D. Bray, *Cell Movements*, Garland, New York, 1992.
- [6] O. Rauprich, M. Matsushita, C.J. Weijer, F. Siegert, S.E. Esipov, and J.A. Shapiro, J. Bacteriol. **178**, 6525 (1996).
- [7] A. Nakahara, Y. Shimada, J. Wakita, M. Matsushita, and T. Matsuyama, J. Phys. Soc. Jap. **65**, 2700 (1996).
- [8] X.-L. Wu and A. Libchaber, Phys. Rev. Lett. **84**, 3017 (2000).
- [9] L. Cisneros, C. Dombrowski, R.E. Goldstein, and J.O. Kessler, Phys. Rev. E **73**, 030901 (2006).
- [10] P.C. Martin, O. Parodi, and P.S. Pershan, Phys. Rev. A **6**, 2401 (1972).
- [11] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions*, (W.A. Benjamin, Reading, Mass. 1975).
- [12] P.G. de Gennes, *The Physics of Liquid Crystals*, (Clarendon Press, Oxford, 1975).
- [13] H. Pleiner and H.R. Brand, in *Pattern Formation in Liquid Crystals*, edited by A. Buka and L. Kramer (Springer, New York, 1996) p. 15.
- [14] R. Graham, Phys. Rev. Lett. **33**, 1431 (1974).
- [15] H.R. Brand, M. Dörfle and R. Graham, Ann. Phys. (N.Y.) **119**, 434 (1979).
- [16] D. Collin, G.K. Auernhammer, O. Gavat, P. Martinoty, and H.R. Brand, Macromol. Rapid Commun. **24**, 737 (2003).
- [17] Z. Varga, J. Feher, G. Filipcsei, and L. Zrinyi, Macromol. Symp. **200**, 93 (2003).
- [18] S. Bohlius, H.R. Brand and H. Pleiner, Phys. Rev. E **70**, 061411 (2004).
- [19] K. Kruse, J.F. Joanny, F. Julicher, J. Prost and K. Sekimoto, Eur. Phys. J. E **16**, 5 (2005).
- [20] S. Muhuri, M. Rao, and S. Ramaswamy, Europhys. Lett. **78**, 48002 (2007).
- [21] D. Svanšek, H. Pleiner, and H.R. Brand, unpublished.
- [22] J.K. Parrish and L. Edelstein-Keshet, Science **284**, 99 (1999).

- [23] J.H. Tien, S.A. Levin, and D. Rubenstein, *Evol. Ecol. Res.* **6**, 555 (2004).
- [24] M. Ballerini et al., *Anim. Behav.* **76**, 201 (2008).
- [25] M. Ballerini et al., *Proc. Nat. Acad. Sci. USA* **105**, 1232 (2008).
- [26] S. Gueron and K. Levit-Gurevich, *Biophys. J.* **74**, 1658 (1998).
- [27] C. Brennen and H. Winet, *Ann. Rev. Fluid Mech.* **9**, 339 (1977).
- [28] D. Bray, *Cell Movements*, Garland, New York, 2nd edition, 2001.
- [29] H. Pleiner, D. Sventšek, and H.R. Brand, unpublished.
- [30] H. Pleiner and H.R. Brand, *Europhys. Lett.* **9**, 243 (1989).
- [31] H.R. Brand, H. Pleiner, and F. Ziebert, *Phys. Rev. E* **74**, 021713 (2006).
- [32] H.R. Brand, P.E. Cladis, and H. Pleiner, *Phys. Rev. E* **79**, 032701 (2009).
- [33] P.G. de Gennes, in *Liquid Crystals of One- and Two-Dimensional Order*, edited by W. Helfrich and G. Heppke (Springer, New York, 1980).
- [34] H.R. Brand and H. Pleiner, *Eur. Phys. J. E* **31**, 37 (2010).
- [35] H.R. Brand and H. Pleiner, *Physica A* **208**, 359 (1994).
- [36] H. Pleiner and H.R. Brand, *Europhys. Lett.* **89**, 26003 (2010).
- [37] H.R. Brand and H. Pleiner, *Phys. Rev. A* **37**, 2736 (1988).
- [38] D. Sventšek, H. Pleiner, and H.R. Brand, *Phys. Rev. Lett.* **96**, 140601 (2006).
- [39] D. Sventšek, H. Pleiner, and H.R. Brand, *Phys. Rev. E* **78**, 021703 (2008).
- [40] Y. Tabe and H. Yokoyama, *Nature Materials* **2**, 806 (2003).
- [41] Y. Tabe, invited talk IL8 at the *European Conference on Liquid Crystals 2007*, ECLC abstract CD-ROM, issued by Universidade de Lisboa, Faculdade de Ciencias et Tecnologia (2007), p.58.
- [42] T.C. Lubensky, *Mol. Cryst. Liq. Cryst.* **23**, 99 (1973).
- [43] M. Liu, *Phys. Rev. Lett.* **43**, 1740 (1979).
- [44] P.E. Cladis, Y. Couder, and H.R. Brand, *Phys. Rev. Lett.* **55**, 2945 (1985).
- [45] P.E. Cladis, P.L. Finn, and H.R. Brand, *Phys. Rev. Lett.* **75**, 1518 (1995).
- [46] O. Lehmann, *Ann. Phys. (Leipzig)* **2**, 649 (1900).